You're Smarter Than You Think

College Algebra for STEM Majors

An Open Educational Resource for Neurodivergent Students

Safaa Dabagh
Santa Monica College • West Los Angeles College

*"You are not behind. You are not broken.
You are not 'bad at math.' That changes now."*

Santa Monica College • West Los Angeles College • 2026

A Note Before We Begin

Something happened in class recently that I want to tell you about.

A student came up to me with their homework. Before they even showed me the page, they said:

> *"I'm pretty sure I'm wrong — but this is what I did."*

I looked at their work.

It was completely correct.

Not almost correct. Not on the right track. **Correct.**

And that moment — that gap between what they believed about themselves and what was actually on the page — is exactly why I wrote this book.

This book grew out of years of teaching College Algebra at **Santa Monica College** and **West Los Angeles College**. The students who inspired every page of it — their questions, their breakthroughs, their courage to keep going — came from both campuses. This book belongs to all of them.

This Book Is For You If:

- You've ever said "I'm not a math person."
- You've stared at a problem and felt your brain go blank before you even started.
- You got the right answer and assumed it was a mistake.
- You understand something in class, then feel like it disappears the moment you walk out the door.
- You've been told you're behind — and started to believe it.
- Your brain works differently, and traditional textbooks don't speak your language.

Here's What I Need You to Know Before You Turn a Single Page:

You are not behind. You are not broken. You are not "bad at math."

You are a person who hasn't yet been shown math in a way that makes sense for how your brain works.

That changes now.

How This Book Works

Every section follows the same rhythm — because your brain learns better when it knows what's coming:

1. **READ THIS FIRST** — A story, a real-life moment that *is* the math concept, before we ever name it.
2. **LET'S TALK ABOUT IT** — A question to think about or discuss. There is no wrong answer here.
3. **NOW WE NAME IT** — We give the concept its official math name, and connect it back to what you already understood.
4. **WATCH IT WORK** — A worked example, every step explained — including the steps most books skip.
5. **YOUR TURN** — You try it. And when your brain says "I'm probably wrong," you keep going anyway.
6. **CHECK YOUR VOICE** — A moment to notice what your inner voice said — and talk back to it.

Contents

Chapter 1

WHO ARE YOU, REALLY?

Sets, Relations, and the Language of Math

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll look at mathematical language and think: *"Oh, I already DO this kind of thinking. I just didn't know the mathematical names for it."*
The Story: You'll see that organizing, connecting, and predicting are things your brain already does — math just gives you precise tools for it.
The Skills You'll Have:

- Recognize and create sets (you do this every day)
- Plot points on the coordinate plane (like giving directions)
- Identify relations and functions (reliable vs unreliable connections)
- Read function notation $f(x)$ without panic

The Confidence Moment: When you see $f(x) = 2x + 1$, your brain will think "function f takes my input, doubles it, and adds 1" instead of "scary math notation I don't understand."
The Bridge: Chapter 2 will show you what functions actually DO — their personalities and behaviors. But first, we need to speak the same language.

1.1 You Already Know What a Set Is

READ THIS FIRST

You wake up and check your phone. You have:

- **Texts from family**
- **Texts from friends**
- **Texts from work**

Without thinking about it, you sort them. Family goes first (or maybe last, depending on your relationship with your family). Work texts get their own mental category. Friend texts might get sorted by urgency or by which friend sent them.
You just did set theory.
A **set** is just a collection of things that belong together according to some rule you decide. Your brain does this automatically, all day long:

- Songs in your "Study Playlist"
- Apps in your "Social Media" folder
- Contacts under "Do Not Answer"
- Food in your "Actually Will Eat This" category

The mathematical word for this is a **set**. But you've been creating and organizing sets since you were old enough to sort your toys into "favorites" and "meh."

LET'S TALK ABOUT IT

Think about how you organize your music, your photos, or even your friends. What are the "rules" you use? How does your brain decide what belongs together?
There's no wrong answer here. Your brain's sorting system is already mathematical — we're just going to give it some official names.

NOW WE NAME IT

In mathematics, we write sets using curly braces: {}
Your "Actually Will Eat This" food category might look like:

{pizza, tacos, anything with cheese, ice cream, mom's cooking}

The **elements** are the individual items inside the set. The **set** is the whole collection.
Sets of numbers work the same way:

- **Natural numbers**: $\{1, 2, 3, 4, 5, \ldots\}$ — the counting numbers
- **Whole numbers**: $\{0, 1, 2, 3, 4, 5, \ldots\}$ — counting numbers plus zero
- **Integers**: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ — positive, negative, and zero
- **Rational numbers**: numbers that can be written as fractions
- **Real numbers**: all numbers that exist on the number line

WATCH IT WORK

Example 1: Is the number -7 in the set of natural numbers?
Step 1: What are the natural numbers? $\{1, 2, 3, 4, 5, \ldots\}$
Step 2: Is -7 in that list? No — natural numbers are positive.
Answer: No, -7 is not a natural number.

Example 2: What set does the number 0.75 belong to?
Step 1: Can we write 0.75 as a fraction? Yes: $0.75 = \frac{3}{4}$
Step 2: Since it can be written as a fraction, it's rational.
Step 3: Since it's rational, it's also real.
Answer: 0.75 belongs to the rational numbers and the real numbers.

YOUR TURN

1. Is the number -12 an integer?
2. What set does the number 2.5 belong to?
3. Create your own set: {things that make you feel calm}

CHECK YOUR VOICE

What did your brain say when you saw those curly braces {} for the first time?
If it said *"This looks complicated"* — that's normal. Curly braces are just punctuation marks, like quotation marks around speech. They're saying: "Here's a collection of things that belong together."
If it said *"I don't know if I'm doing this right"* — remember, you've been making sets your whole life. You're just learning the mathematical way to write them down.

1.2 The Cartesian Plane (Your Mental Map)

READ THIS FIRST

You're meeting a friend at a new restaurant. They text you:

> *"I'm at the table by the window, three tables back from the front, second table from the left wall."*

You walk in and find them immediately.
You just used a **coordinate system**.
Your friend gave you two pieces of information:

1. **How far back** from the front (like a y-coordinate)
2. **How far over** from the left wall (like an x-coordinate)

Those two numbers together pinpoint exactly one location in the restaurant.
The **Cartesian plane** works exactly the same way. It's just a mathematical restaurant where every point has an address made of two numbers: (x, y).

- The x tells you how far left or right from center
- The y tells you how far up or down from center
- The center point is $(0, 0)$ — like the entrance to our mathematical restaurant

LET'S TALK ABOUT IT

Think about other "address systems" you use daily:

- Your seat at a movie theater (Row F, Seat 12)
- A cell on a spreadsheet (Column C, Row 5)
- Your location on a map app (latitude, longitude)

How are these similar to what your friend did with the restaurant directions?

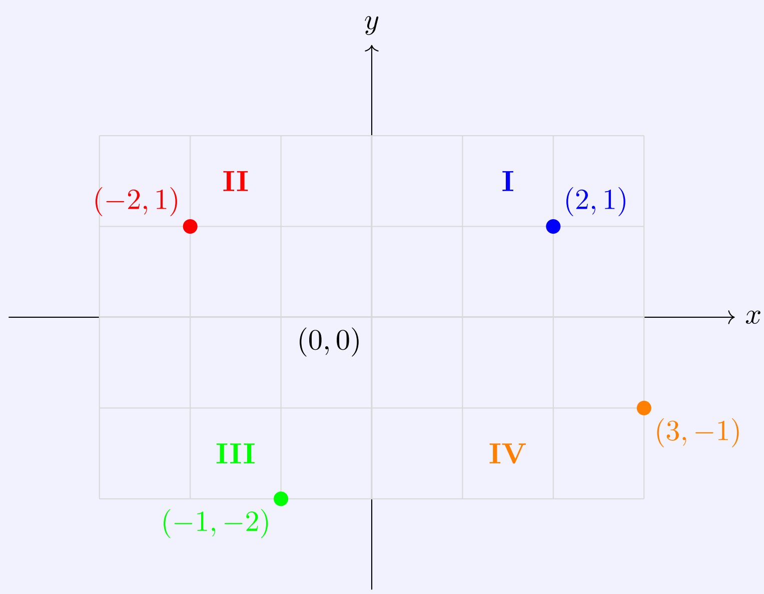
NOW WE NAME IT

The **Cartesian plane** (named after René Descartes) has:

- **Two perpendicular axes** (fancy word for "number lines")
- The x-axis runs horizontally (left-right)
- The y-axis runs vertically (up-down)
- Every point has coordinates (x, y)

Quadrants are like neighborhoods:

- **Quadrant I**: x positive, y positive (upper right)
- **Quadrant II**: x negative, y positive (upper left)
- **Quadrant III**: x negative, y negative (lower left)
- **Quadrant IV**: x positive, y negative (lower right)



WATCH IT WORK

Example 1: Plot the point $(3, -2)$
Step 1: Start at the origin $(0, 0)$
Step 2: Move 3 units right (because $x = 3$)
Step 3: Move 2 units down (because $y = -2$)
Step 4: Mark the spot — that's your point!

Example 2: What quadrant is the point $(-4, 5)$ in?
Step 1: Look at the signs: $x = -4$ (negative), $y = 5$ (positive)
Step 2: Negative x, positive y means upper left
Answer: Quadrant II

▷ DESMOS EXPLORATION `desmos.com/calculator`

1. Go to `desmos.com/calculator`
2. Click the + button and choose **Table**
3. Enter x_1 values: `2, -1, 0, -3` and y_1 values: `3, 4, -3, -2`
4. Watch all four points appear instantly on the coordinate plane
5. **Click and drag** any point — the coordinates update live!
6. Challenge: drag a point into each quadrant and name the quadrant before releasing

Every point has exactly one address — just like every seat in a stadium has one row and column number.

YOUR TURN

1. Plot these points: $(2, 3)$, $(-1, 4)$, $(0, -3)$
2. What quadrant is each point in?
3. Find a point in Quadrant III and write its coordinates

CHECK YOUR VOICE

What did your brain say when you saw "Cartesian plane"?
If it said *"That sounds intimidating"* — remember, it's just a fancy name for "mathematical restaurant seating chart." Descartes was just a guy who had a good idea about organizing space.
If it said *"I always mix up x and y"* — think "x across" (both have the letter 'x') and "y up to the sky." Your brain likes patterns, and this one will stick.

1.3 Relations (When Things Connect)

READ THIS FIRST

Open your contacts list on your phone.
Look at any name. Let's say... "Alex."
Now ask yourself: *What's Alex's relationship to you?*
Maybe:

- Alex → college roommate
- Alex → cousin
- Alex → person from work
- Alex → "honestly I have no idea but they're in my phone"

Your brain just created a **relation**. You took one thing (Alex) and connected it to another thing (roommate/cousin/coworker).
In mathematics, a **relation** is exactly this: a way that things in one set connect to things in another set.
Your phone is full of relations:

- **Person** → **Job title**
- **Contact** → **Phone number**
- **Friend** → **Favorite coffee order** (if you're a good friend)
- **Song** → **Mood** (when you're making playlists)

Every time your brain says "this goes with that," you're thinking relationally. Mathematics just gives us tools to be more precise about it.

LET'S TALK ABOUT IT

Think about the relationships in your life:

- Students in your class → Their major
- Movies → Your star rating
- Days of the week → Your energy level

What makes a good relationship? What makes a confusing one?

NOW WE NAME IT

A **relation** connects elements from one set (called the **domain**) to elements in another set (called the **range**).

We can show relations in several ways:

1. As ordered pairs: $\{(\text{Alex, roommate}), (\text{Sam, cousin}), (\text{Jordan, coworker})\}$

2. As a table:

Person	**Relationship**
Alex	roommate
Sam	cousin
Jordan	coworker

3. As a mapping diagram: (arrows from one list to another)

4. As a graph: points plotted on the coordinate plane

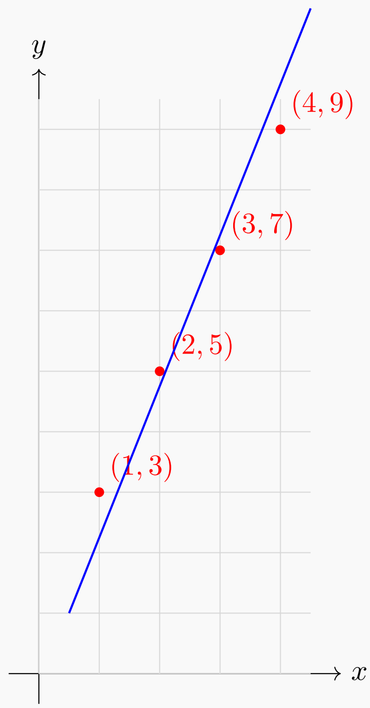
WATCH IT WORK

Example 1: Here's a relation showing students and their favorite subjects:

$$\{(\text{Maria, Biology}), (\text{James, Physics}), (\text{Sarah, Math}), (\text{Alex, Biology})\}$$

Domain: {Maria, James, Sarah, Alex}
Range: {Biology, Physics, Math}

Example 2: Graph the relation $\{(1,3),(2,5),(3,7),(4,9)\}$
Step 1: Plot each point on the coordinate plane
Step 2: Notice the pattern — each y-value is 2 more than twice the x-value
Step 3: This relation has a nice, predictable pattern



▷ DESMOS EXPLORATION `desmos.com/calculator`

1. Open `desmos.com/calculator`
2. Type each ordered pair on its own line: `(1,3)`, `(2,5)`, `(3,7)`, `(4,9)`
3. Type `x = 1` — this draws a vertical line at $x = 1$
4. Does the vertical line hit more than one point? **No** ⇒ this could be a function
5. Now add the point `(1, 8)` and re-run the test. What changes?

The Vertical Line Test is the definition of a function in picture form: one input, exactly one output.

YOUR TURN

1. Create a relation: "Day of the week → How much coffee you need"
2. Write it as ordered pairs
3. What's the domain? What's the range?

CHECK YOUR VOICE

What happened in your brain when you saw "domain" and "range"?
If it said *"More vocabulary to memorize"* — remember, these are just names for things you already understand. Domain = the starting set, Range = the ending set. Like "departures" and "arrivals" at an airport.
If it said *"This seems too simple"* — that's because relations ARE simple when you think of them as connections your brain already makes. We're just organizing your natural thinking.

1.4 Functions (The Reliable Friend)

READ THIS FIRST

You have that one friend who's completely predictable.
Ask them "What's your favorite pizza topping?" and they'll always say "Pepperoni."
Ask them next week, they'll say "Pepperoni."
Ask them next month, same answer: "Pepperoni."
Ask them what they want to watch on Netflix? "Something with action." Every time.
This friend never gives you a different answer to the same question. They're **reliable**. Maybe a little boring, but reliable.
A **function** is mathematics' version of your reliable friend.
Give a function the same input, and it will **always** give you the same output. Every single time. No exceptions. No mood swings. No "it depends on how I'm feeling today."

- Input 3 $\to$ Output 9 (every time)
- Input 5 $\to$ Output 25 (every time)
- Input -2 $\to$ Output 4 (every time)

That's what makes it a function: **one input = exactly one output, always.**

LET'S TALK ABOUT IT

Think about the "reliable friends" in your life — people, apps, or systems that always respond the same way to the same situation:

- Your coffee shop barista who remembers your order
- Your car that starts the same way every time
- Your favorite playlist that always puts you in the right mood

What makes these relationships reliable? What would happen if they suddenly became unpredictable?

NOW WE NAME IT

A **function** is a special type of relation where:

- Every input has exactly one output
- No input gets multiple outputs (that would be unreliable!)

Function notation: $f(x) =$ "what function f does to input x"
NOT "f times x" — it's "function f applied to x" or "what f gives you when you put in x"
Think of $f(x)$ like:

- f = the machine/friend
- x = what you give them
- $f(x)$ = what they give you back

The Vertical Line Test: If you can draw a vertical line that crosses a graph more than once, it's not a function (because one input would have multiple outputs).

WATCH IT WORK

Example 1: Is this relation a function?

$$\{(1,3),(2,6),(3,9),(4,12)\}$$

Step 1: Check each input — does it appear more than once?
Step 2: Input 1 appears once, 2 appears once, 3 appears once, 4 appears once
Answer: Yes, it's a function! Each input has exactly one output.

Example 2: If $f(x) = 2x + 1$, find $f(4)$
Step 1: Substitute 4 for x: $f(4) = 2(4) + 1$
Step 2: Calculate: $f(4) = 8 + 1 = 9$
Answer: $f(4) = 9$

Example 3: Is the relation "person → birthday" a function?
Step 1: Can one person have multiple birthdays? No.
Step 2: Does each person have exactly one birthday? Yes.
Answer: Yes, it's a function!

YOUR TURN

1. Is $\{(2,4),(3,9),(2,8)\}$ a function? Why or why not?
2. If $g(x) = x^2 - 3$, find $g(5)$
3. Give an example from your life of something that IS a function
4. Give an example of something that is NOT a function

CHECK YOUR VOICE

What did your brain say when you saw $f(x)$ for the first time?
If it said *"I don't understand function notation"* — remember, $f(x)$ is just a way of writing "what function f does to x." Like writing "Alex's phone number" instead of just "phone number." The parentheses aren't multiplication — they're just saying "f applied to x."
If it said *"Functions seem abstract"* — they're actually the opposite! Functions are everywhere: your age as a function of time, your phone battery as a function of usage, your mood as a function of how much sleep you got. You interact with functions all day long.
If it said *"I think I actually get this"* — THAT'S THE VOICE WE WANT TO HEAR! You've been understanding functions your whole life. Now you just know what they're called.

YOUR TURN - EXTENDED PRACTICE

Chapter 1: Building Your Mathematical Confidence

Problems That Connect the Pieces

1. Set Creation (Your Brain Already Does This)
Create three sets from your own life:

a) {Apps on your phone that you actually use}

b) {Classes this semester → your predicted grade}

c) {Friends → their go-to coffee order}

Which of these is a function? How do you know?

2. Coordinate Confidence
Plot these points and describe the story they tell:

$$(0, 32), (10, 50), (20, 68), (30, 86), (40, 104)$$

Hint: These points show temperature conversion. What do you notice about the pattern?

3. Function Detective Work
Look at this data about students and their study hours:

$$\{(\text{Maria}, 3 \text{ hours}), (\text{James}, 5 \text{ hours}), (\text{Sarah}, 2 \text{ hours}), (\text{Maria}, 4 \text{ hours})\}$$

a) Is this a function? Why or why not?

b) How could you fix it to make it a function?

c) What does this tell you about collecting data in real life?

4. Confidence Check
Write a short paragraph explaining functions to a friend who's never taken algebra. Use your own words — no mathematical jargon allowed.

5. Building Forward
You now know that $f(x) = 2x + 3$ means "double the input and add 3."
Predict what these functions do in plain English:

a) $g(x) = x + 5$

b) $h(x) = 3x$

c) $k(x) = x^2$

We'll explore these personalities in Chapter 2!

ANSWER KEY - CHAPTER 1

Building Your Mathematical Confidence

Checking Your Answers (And Celebrating Your Success!)

Problem 1: Set Creation
Your answers will be unique to you — that's the beauty of this problem! Here's what to check:

a) {Apps you actually use} — This is just a set, not a function

b) {Classes → predicted grades} — This IS a function! Each class gets exactly one predicted grade

c) {Friends → coffee orders} — This IS a function! Each friend has exactly one go-to order

Key insight: You created functions without even thinking about it! Your brain naturally organizes information this way.

Problem 2: Coordinate Confidence
These points show **Celsius to Fahrenheit conversion**! - $(0, 32)$ means 0°C = 32°F (water freezes) - $(10, 50)$ means 10°C = 50°F - $(20, 68)$ means 20°C = 68°F (room temperature) - $(30, 86)$ means 30°C = 86°F - $(40, 104)$ means 40°C = 104°F (hot summer day)
Pattern: For every 10°C increase, Fahrenheit increases by 18°F
If you noticed ANY pattern, you're thinking like a mathematician!

Problem 3: Function Detective Work

a) **Not a function!** Maria appears twice with different study hours (3 and 4). One input, two outputs = not a function.

b) **Fix it by:**
   - Average Maria's hours: $(\text{Maria}, 3.5 \text{ hours})$
   - Choose the more recent data: $(\text{Maria}, 4 \text{ hours})$
   - Specify which week: $(\text{Maria Week } 1, 3 \text{ hours}), (\text{Maria Week } 2, 4 \text{ hours})$

c) **Real life insight:** Data collection is messy! Functions help us organize and make sense of information.

Problem 4: Confidence Check
There's no single "right" answer here! If you mentioned that functions are reliable, predictable, or like machines that always give the same output for the same input, you nailed it!
Great explanations might include:

- "A function is like a reliable friend who always responds the same way"
- "It's a rule that turns one number into exactly one other number"
- "Think of it like a vending machine — same button always gives same snack"

If you wrote something and it makes sense to you, it's probably a good explanation!

CHECK YOUR VOICE - AFTER ANSWERS

What did your brain say when you checked your answers?
If it said *"I got most of these wrong"* — remember, this isn't about being right or wrong. It's about building understanding. Every "mistake" is actually your brain learning!
If it said *"These seemed too easy"* — that means the explanations are working! Math should make sense, not feel impossible.
If it said *"I actually understood more than I thought"* — EXACTLY! You've been doing mathematical thinking your whole life. We just gave it official names.
Ready for Chapter 2? You're about to discover that functions have personalities, and you're going to love getting to know them!

Chapter 2

FUNCTIONS — GETTING TO KNOW THEM

SKILLS CHECK: What You Need From Chapter 1

- **Sets & Elements:** We'll work with domain and range (the input and output sets)
- **Coordinate Plane:** We'll graph functions to see their personalities visually
- **Function Basics:** We'll build on $f(x)$ notation and add arithmetic operations
- **The "Reliable Friend" Concept:** Functions always give the same output for the same input

If any of these feel shaky, flip back to Chapter 1 for a quick review!

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll think of functions as **mathematical personalities** — each with their own behavior, style, and predictable patterns.
The Story: You'll meet the Function Machine and learn that $f(x)$ isn't scary notation — it's just a way to describe what happens when you feed something into a reliable system.
The Skills You'll Have:

- Read and write function notation fluently
- Add, subtract, multiply, and divide functions (like mixing personalities)
- Understand domain and range as natural boundaries
- Graph functions and see their visual personalities
- Transform functions (stretch, shift, flip) and predict the results

The Confidence Moment: When you see $f(x) = 3x - 7$, your brain will think "Oh, the tripler-and-subtract-7 function!" instead of panicking about notation.
The Bridge: Chapter 3 will introduce specific function families (linear, quadratic, etc.) — but first, you need to be comfortable with how functions behave in general.

2.1 The Function Machine (What Does f(x) Actually Mean?)

READ THIS FIRST

You walk into your favorite coffee shop. You know exactly what's going to happen.
You say: "Medium latte with oat milk."
The barista: *types into the register* → *makes your drink* → hands you the same perfect latte you always get.
You just used a **function machine**.

- **Your input:** "Medium latte with oat milk"
- **The machine:** Trained barista + espresso equipment + process
- **Your output:** One medium latte with oat milk (every time)

This is exactly what $f(x)$ means:

- f = the machine (barista + equipment + process)
- x = your input ("medium latte with oat milk")
- $f(x)$ = what the machine gives you back (your latte)

The notation $f(x) = 2x + 1$ is just saying: *"Here's a machine called f. When you put in x, it doubles it and adds 1."*
Not "f times x." Not multiplication at all. Just: "Here's what this particular machine does to whatever you give it."

LET'S TALK ABOUT IT

Think about other "function machines" in your life:

- Vending machine: input money + button → output snack
- Your car: input key turn + gas pedal → output movement
- Your streaming service: input "crime documentaries" → output recommendations

What makes these machines reliable? What would happen if they started giving random outputs?

NOW WE NAME IT

Function notation tells you three things:

1. **The name of the machine:** f, g, h, etc.
2. **What you're putting in:** the variable (usually x)
3. **What the machine does:** the rule/formula

Examples:

- $f(x) = x + 5 \rightarrow$ "Machine f adds 5 to whatever you put in"
- $g(x) = 3x \rightarrow$ "Machine g triples whatever you put in"
- $h(x) = x^2 \rightarrow$ "Machine h squares whatever you put in"

Domain: The set of all possible inputs (what the machine can accept)
Range: The set of all possible outputs (what the machine can produce)

WATCH IT WORK

Example 1: If $f(x) = 2x + 3$, find $f(5)$
Step 1: Replace every x with 5: $f(5) = 2(5) + 3$
Step 2: Follow the machine's instructions: $f(5) = 10 + 3$
Step 3: Get the output: $f(5) = 13$
Translation: "When I put 5 into machine f, I get 13."

Example 2: If $g(x) = x^2 - 4$, find $g(-3)$
Step 1: Replace every x with -3: $g(-3) = (-3)^2 - 4$
Step 2: Follow the machine's instructions: $g(-3) = 9 - 4$
Step 3: Get the output: $g(-3) = 5$

Example 3: What does the function $f(x) = -x + 7$ do in plain English?
Translation: "Machine f takes your input, makes it negative, then adds 7."

YOUR TURN

1. If $f(x) = 4x - 1$, find:
   a) $f(3)$
   b) $f(0)$
   c) $f(-2)$
2. If $g(x) = x^2 + 2x$, find $g(4)$
3. Describe these functions in plain English:
   a) $h(x) = \frac{x}{2}$
   b) $k(x) = x + 10$
   c) $m(x) = 5x$

CHECK YOUR VOICE

What did your brain say when you worked with $f(5) = 2(5) + 3$?
If it said *"This is just substitution"* — EXACTLY! You've been doing this kind of thinking for years. Function notation is just an organized way to write it.
If it said *"I keep wanting to multiply f times x"* — that's normal! Your brain sees letters next to each other and thinks multiplication. But $f(x)$ is like writing "Alex's phone number" — the parentheses show possession/application, not multiplication.
If it said *"This is actually kind of logical"* — YES! That's exactly the shift we want. Functions are logical systems, not mysterious notation.

2.2 Function Arithmetic (When Machines Collaborate)

READ THIS FIRST

You and your friend both work at different coffee shops in town.
Your coffee shop function: $f(x) =$ "Make x cups of coffee"
Your friend's shop function: $g(x) =$ "Make x cups of tea"
Now imagine you're both catering the same event together. The event organizer asks:
"Can you make 20 cups of coffee AND 20 cups of tea?"
You're basically computing: $f(20) + g(20)$

- Your shop makes: $f(20) = 20$ cups of coffee
- Friend's shop makes: $g(20) = 20$ cups of tea
- Together you make: 20 cups of coffee + 20 cups of tea = 40 total beverages

You just did **function addition**: $(f + g)(20) = f(20) + g(20)$
Functions can work together just like people can. They can:

- **Add their outputs** $(f + g)$
- **Subtract their outputs** $(f - g)$
- **Multiply their outputs** $(f \cdot g)$
- **Divide their outputs** $(f \div g)$

Each combination creates a new collaborative function with its own personality.

LET'S TALK ABOUT IT

Think about times when you combine different processes:

- Your morning routine: shower function + coffee function + getting dressed function = ready for the day function
- Cooking: prep function + cooking function + plating function = complete meal function
- Your budget: income function - expenses function = savings function

How do these combinations create something new?

NOW WE NAME IT

When you have two functions $f(x)$ and $g(x)$, you can combine them:
$(f+g)(x) = f(x) + g(x) \rightarrow$ "Add the outputs"
$(f-g)(x) = f(x) - g(x) \rightarrow$ "Subtract the outputs"
$(f \cdot g)(x) = f(x) \cdot g(x) \rightarrow$ "Multiply the outputs"
$(f \div g)(x) = \frac{f(x)}{g(x)} \rightarrow$ "Divide the outputs" (when $g(x) \neq 0$)
Domain of combined functions: Usually the intersection of both original domains (where both functions can operate).

WATCH IT WORK

Example 1: If $f(x) = 2x + 1$ and $g(x) = x - 3$, find $(f + g)(4)$
Method 1 - Find each output first:

$$f(4) = 2(4) + 1 = 9 \tag{2.1}$$
$$g(4) = 4 - 3 = 1 \tag{2.2}$$
$$(f+g)(4) = f(4) + g(4) = 9 + 1 = 10 \tag{2.3}$$

Method 2 - Combine the functions first:

$$(f+g)(x) = f(x) + g(x) = (2x+1) + (x-3) = 3x - 2 \tag{2.4}$$
$$(f+g)(4) = 3(4) - 2 = 10 \tag{2.5}$$

Both methods give the same answer!

Example 2: If $f(x) = x^2$ and $g(x) = 2x$, find $(f \cdot g)(3)$
Step 1: Find each output: $f(3) = 3^2 = 9$, $g(3) = 2(3) = 6$
Step 2: Multiply them: $(f \cdot g)(3) = f(3) \cdot g(3) = 9 \cdot 6 = 54$

Example 3: Create the general formula for $(f - g)(x)$ if $f(x) = 3x + 5$ and $g(x) = x + 2$
Step 1: Write the combination: $(f - g)(x) = f(x) - g(x)$
Step 2: Substitute: $(f - g)(x) = (3x + 5) - (x + 2)$
Step 3: Simplify: $(f - g)(x) = 3x + 5 - x - 2 = 2x + 3$

YOUR TURN

1. If $f(x) = x + 4$ and $g(x) = 2x - 1$, find:
   a) $(f + g)(2)$
   b) $(f - g)(5)$
   c) $(f \cdot g)(1)$
2. If $h(x) = x^2$ and $k(x) = x + 3$, find $(h \div k)(2)$
3. Create the general formula for $(f + g)(x)$ if $f(x) = 4x$ and $g(x) = x^2$

CHECK YOUR VOICE

What did your brain think when you saw $(f+g)(x)$?

If it said *"This looks like more complicated notation"* — remember, it's just shorthand for "add the outputs of two machines." Like saying "team up" instead of listing everyone's individual contributions.

If it said *"I mixed up the order of operations"* — that's common! When you see $(f+g)(4)$, work from the inside out: first find $f(4)$ and $g(4)$, then add them. The parentheses are your guide.

If it said *"This is actually pretty logical"* — EXACTLY! You're combining simple processes to create more complex ones. It's like following a recipe that combines multiple cooking techniques.

2.3 Domain and Range (The Function's Boundaries)

READ THIS FIRST

Your favorite streaming service has recommendations for you.
But notice: it doesn't recommend *everything*.
It can only recommend:

- **Movies that actually exist** (not ones you made up in your head)
- **Movies available in your region** (not ones blocked in your country)
- **Movies in languages/genres it has data for** (not super obscure films it's never heard of)

Those limitations are your streaming service's **domain** — the set of inputs it can actually work with.
And the recommendations it gives you are limited too:

- **Only movies in its catalog** (it can't recommend Netflix shows if you're on Hulu)
- **Only ratings/genres it tracks** (it won't recommend "movies that smell good")

Those possible outputs are its **range**.
Every function has natural boundaries:

- **Domain:** What inputs make sense/are allowed
- **Range:** What outputs are actually possible

Your brain already understands this — you don't ask your coffee shop for sushi, and you don't ask your math teacher for dating advice. You intuitively know the domain and range of different "life functions."

LET'S TALK ABOUT IT

Think about the "domains" of different services or people in your life:

- Your bank ATM: what inputs does it accept? What can't it do?
- Your car GPS: what destinations can it handle? What would break it?
- Your best friend's advice: what topics are they good with? Where are their limits?

What happens when you try to use something outside its domain?

NOW WE NAME IT

Domain: The set of all possible input values (x-values) that make the function work
Range: The set of all possible output values (y-values) the function can produce
Common domain restrictions:

- **Division by zero:** Can't divide by zero
- **Square roots:** Can't take square root of negative numbers (in real numbers)
- **Real-world limits:** Age can't be negative, you can't have half a person

Finding domain: Ask "What values of x would break this function?"
Finding range: Ask "What outputs are actually possible?"

WATCH IT WORK

Example 1: Find the domain of $f(x) = 3x + 7$
Step 1: Ask "What would break this function?"
Step 2: We can multiply any number by 3, and add 7 to anything
Step 3: Nothing breaks this function!
Domain: All real numbers

Example 2: Find the domain of $g(x) = \frac{1}{x-2}$
Step 1: Ask "What would break this function?"
Step 2: Division by zero would break it
Step 3: When does the denominator equal zero? When $x - 2 = 0$, so $x = 2$
Domain: All real numbers except $x = 2$

Example 3: A function represents "profit from selling x items" where each item costs \$5 to make and sells for \$12.
Function: $P(x) = 12x - 5x = 7x$
Domain consideration: Can you sell negative items? No. Can you sell half an item? Depends on the business.
Practical domain: $x \geq 0$, and maybe only whole numbers

▷ **DESMOS EXPLORATION** desmos.com/calculator

1. Type `y = 1/(x-2)` in Desmos — notice the gap at $x = 2$. That gap *is* the domain restriction.
2. Type `y = sqrt(x-1)` — observe exactly where the graph starts. Why can't it go left of $x = 1$?
3. Click the left endpoint of the graph. Read the coordinates.
4. **Challenge:** Type `y = 1/sqrt(x-3)` — predict the domain *before* looking at the graph, then check.

Desmos enforces domain restrictions automatically. The gap or cutoff you see on screen is *the domain restriction made visible.*

YOUR TURN

1. Find the domain of each function:
   a) $f(x) = x^2 + 4$
   b) $g(x) = \frac{1}{x+3}$
   c) $h(x) = \sqrt{x-1}$
2. A function represents the number of hours you sleep based on how many cups of coffee you drink: $S(c) = 8 - 2c$. What's a reasonable domain for this function?
3. If $f(x) = x - 5$, what's the range of this function?

CHECK YOUR VOICE

What did your brain do when asked to find a domain?
If it said *"I don't know where to start"* — start by asking "What would make this function impossible or undefined?" Usually it's division by zero or square roots of negative numbers.
If it said *"The real-world context seems harder than the math"* — that's actually good thinking! Real-world domains require you to think about what makes practical sense, not just what's mathematically possible.
If it said *"This is just logical reasoning"* — YES! Domain and range are about understanding limitations and possibilities. You do this kind of reasoning all the time.

2.4 Function Graphs (Seeing the Personality)

READ THIS FIRST

You're scrolling through social media and you see two posts:
Post 1: A wall of text describing someone's vacation
Post 2: A photo from that same vacation
Which one do you look at first? Which one tells you the story faster?
The photo, right? Your brain processes visual information incredibly quickly. One glance and you know: beach vacation, sunny day, looks fun.
Function graphs are the "photos" of mathematics.
Instead of staring at $f(x) = 2x + 1$ and trying to imagine what it does, you can look at its graph and immediately see:

- It's a straight line
- It's going upward
- It crosses the y-axis at 1
- For every 1 step right, it goes 2 steps up

Your brain gets the whole personality of the function in one visual snapshot.
Every function has a visual personality that shows up when you graph it. Some are straight lines (the steady, predictable types). Others are curves (the more dramatic personalities). Some have sharp corners (the functions that make sudden decisions).

LET'S TALK ABOUT IT

Think about how visuals help you understand things:

- Weather apps: you see the week at a glance instead of reading "Monday 75°, Tuesday 73°, Wednesday 71°..."
- Your fitness app: you see your progress over time as a line going up (hopefully!)
- Your bank app: you see spending patterns as charts

How do these visuals change your understanding compared to just reading numbers?

NOW WE NAME IT

A **graph of a function** shows:

- x**-axis:** All possible inputs (domain)
- y**-axis:** All possible outputs (range)
- **The curve/line:** The relationship between inputs and outputs

Key features to notice:

- y**-intercept:** Where the graph crosses the y-axis (when $x = 0$)
- x**-intercept(s):** Where the graph crosses the x-axis (when $y = 0$)
- **Shape:** Line, curve, sharp corners, smooth bends
- **Direction:** Going up, going down, or both
- **Steepness:** Gentle slope or dramatic changes

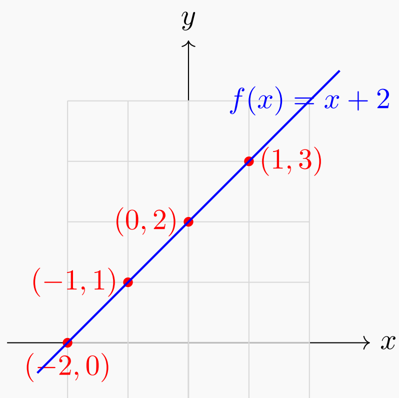
Reading the graph: Every point (x, y) tells you "when input $= x$, output $= y$"

WATCH IT WORK

Example 1: Graph $f(x) = x + 2$
Step 1: Make a table of values:

x	$f(x) = x + 2$	Point
-2	$f(-2) = -2 + 2 = 0$	$(-2, 0)$
-1	$f(-1) = -1 + 2 = 1$	$(-1, 1)$
0	$f(0) = 0 + 2 = 2$	$(0, 2)$
1	$f(1) = 1 + 2 = 3$	$(1, 3)$

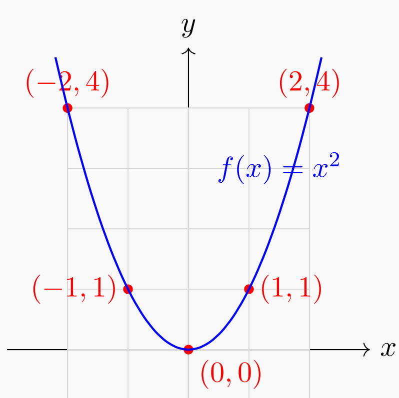
Step 2: Plot the points and connect them
Step 3: Notice: it's a straight line going up!



Example 2: What does the graph of $f(x) = x^2$ look like?
Make a table:

x	$f(x) = x^2$	Point
-2	$f(-2) = (-2)^2 = 4$	$(-2, 4)$
-1	$f(-1) = (-1)^2 = 1$	$(-1, 1)$
0	$f(0) = 0^2 = 0$	$(0, 0)$
1	$f(1) = 1^2 = 1$	$(1, 1)$
2	$f(2) = 2^2 = 4$	$(2, 4)$

Result: A U-shaped curve called a parabola!



Example 3: Reading a graph
If you see a point $(3, 7)$ on the graph of function g, what does that tell you?

YOUR TURN

1. Make a table of values and sketch $f(x) = 2x$:

   Use x values: $-2, -1, 0, 1, 2$

2. For the function $g(x) = x^2 - 1$, find these points:

   a) $g(0) = ? \rightarrow$ point (0, ?)

   b) $g(2) = ? \rightarrow$ point (2, ?)

   c) $g(-2) = ? \rightarrow$ point (-2, ?)

3. If you see the point $(4, 10)$ on a graph of function h, what is $h(4)$?

CHECK YOUR VOICE

What did your brain say when you first saw a function graph?

If it said *"This is just plotting points"* — exactly! You've been doing this for years. A function graph is just an organized way of showing lots of input-output pairs at once.

If it said *"I can't tell what the function is from looking at the graph"* — that's actually advanced thinking! Reading graphs fluently takes practice, but you're already noticing that the visual and the equation are connected.

If it said *"I can see the pattern better this way"* — YES! That's the power of graphing. Your brain can spot patterns visually much faster than by staring at equations.

YOUR TURN - EXTENDED PRACTICE

Chapter 2: Building Function Fluency

Problems That Connect All the Pieces

1. Function Machine Mastery
You run three different "function machines" at a math carnival:

- Machine A: $f(x) = 3x - 2$
- Machine B: $g(x) = x^2 + 1$
- Machine C: $h(x) = -2x + 5$

a) If a customer puts 4 into each machine, what comes out?

b) Which machine gives the biggest output for input 4?

c) Describe each machine's personality in plain English.

2. Function Collaboration Challenge
Using the same machines from Problem 1:

a) What happens if you run machines A and C together? Find $(f + h)(3)$.

b) What if machine A's output becomes machine B's input? Find $g(f(2))$.

c) Create a new machine that subtracts B's output from A's output. What's the formula?

3. Domain Detective
Determine the domain for each function and explain your reasoning:

a) $f(x) = x + 7$ (the "add 7" machine)

b) $g(x) = \frac{1}{x-4}$ (the "divide by $x - 4$" machine)

c) $h(x) = \sqrt{x+3}$ (the "square root" machine)

d) Real-world function: $C(t)$ = cost of parking for t hours, where the garage closes at night

4. Graph Story-Telling
Sketch these functions and describe their visual personalities:

a) $f(x) = x$ (the "identity" function - gives back what you put in)

b) $g(x) = -x$ (the "opposite" function)

c) $h(x) = x + 3$ (how is this different from the identity function?)

5. Function in the Wild
Your phone battery percentage as a function of hours since morning:

$$B(t) = 100 - 8t \text{ (assuming 8\% drain per hour)}$$

ANSWER KEY - CHAPTER 2

Building Function Fluency

Checking Your Answers (And Celebrating Your Understanding!)

Problem 1: Function Machine Mastery

a) **Input 4 into each machine:**

- Machine A: $f(4) = 3(4) - 2 = 12 - 2 = 10$
- Machine B: $g(4) = 4^2 + 1 = 16 + 1 = 17$
- Machine C: $h(4) = -2(4) + 5 = -8 + 5 = -3$

b) **Biggest output:** Machine B gives 17 (the winner!)

c) **Machine personalities:**

- Machine A: "The triple-and-subtract-2 machine"
- Machine B: "The square-and-add-1 machine" (dramatic growth!)
- Machine C: "The double-negative-and-add-5 machine" (contrarian personality)

Key insight: You described mathematical operations as personalities! That's exactly how mathematicians think about functions.

Problem 2: Function Collaboration Challenge

a) **Machines A and C together:** $(f + h)(3)$

$$f(3) = 3(3) - 2 = 7 \tag{2.6}$$
$$h(3) = -2(3) + 5 = -1 \tag{2.7}$$
$$(f + h)(3) = 7 + (-1) = 6 \tag{2.8}$$

b) **A's output into B:** $g(f(2))$

$$f(2) = 3(2) - 2 = 4 \tag{2.9}$$
$$g(f(2)) = g(4) = 4^2 + 1 = 17 \tag{2.10}$$

This is called **function composition** — we'll explore this more later!

c) **A minus B:** $(f-g)(x) = f(x)-g(x) = (3x-2)-(x^2+1) = 3x-2-x^2-1 = -x^2+3x-3$

Problem 3: Domain Detective

a) $f(x) = x + 7$: **All real numbers** (nothing breaks this!)

b) $g(x) = \frac{1}{x-4}$: **All real numbers except** $x = 4$ (can't divide by zero)

c) $h(x) = \sqrt{x+3}$: $x \geq -3$ (can't take square root of negative numbers)

d) $C(t)$ parking cost: $0 \leq t \leq 12$ **hours** (depends on garage hours!)

Great detective work! You're thinking about what would break each function.

CHECK YOUR VOICE - AFTER CHAPTER 2

What did your brain say as you worked through Chapter 2?
If it said *"Function notation is starting to make sense"* — YES! That's exactly what we wanted. $f(x)$ isn't scary anymore — it's just organized thinking.
If it said *"I can actually see patterns in the graphs"* — Your visual brain is connecting with the mathematical brain. This is powerful!
If it said *"These functions do have personalities"* — You've discovered something mathematicians love: functions aren't just formulas, they're characters with behaviors and stories.
If it said *"I think I'm getting good at this"* — LISTEN TO THAT VOICE! You're building mathematical confidence, and that confidence is well-earned.
Ready for Chapter 3? We're about to meet the Linear Family (the steady, reliable lines) and the Quadratic Family (the dramatic curves). Each family has its own personality, and you're going to recognize them like old friends!
You're not just learning math — you're discovering that you've been mathematical all along.

CHAPTER 2 COMPLETE!

You now think in terms of function machines, you can combine functions like collaborative teams, and you can read the visual personality of any function graph.
Most importantly: When you see $f(x)$, your brain now thinks "function f applied to x" instead of "scary math notation."
Ready for Chapter 3? We're about to meet the function families — Linear (the steady one), Quadratic (the dramatic curve), and Absolute Value (the optimist who's always positive). Each has their own personality, their own story, and their own way of solving real-world problems.
The mathematical revolution continues!

Chapter 3

LINES, CURVES & THE DRAMA OF QUADRATICS

Linear, Absolute Value & Quadratic Functions

SKILLS CHECK: What You Need From Previous Chapters

- **Coordinate Plane:** We'll plot lots of function families and see their personalities
- **Function Notation:** We'll work with $f(x) = mx + b$, $g(x) = |x|$, and $h(x) = ax^2 + bx + c$
- **Function Graphs:** We'll compare how different families look and behave
- **Domain and Range:** Each family has its own natural boundaries and outputs

If any of these feel shaky, flip back to Chapters 1-2 for a quick review!

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll think of functions as **family members** with distinct personalities - some steady and predictable, others dramatic and curved.
The Story: You'll finally understand why someone would write 2 as $\frac{2}{1}$ (it tells a richer story), meet the reliable Linear family, the optimistic Absolute Value family, and the dramatic Quadratic family.
The Skills You'll Have:

- Recognize linear functions by their steady, straight-line personality
- Understand slope as a story about how two things change together
- Work with absolute value functions (the eternal optimists)
- Master quadratic functions and their U-shaped drama
- Predict function behavior just by looking at equations

The Confidence Moment: When you see $f(x) = 3x - 7$, your brain will think "Oh, it's a line with slope 3, starting at -7" instead of "confusing algebra stuff."
The Bridge: Chapter 4 will show you what happens when you multiply these families together to create more complex polynomial personalities.

3.1 The Linear Family (Steady and Reliable)

READ THIS FIRST

Let me tell you about that moment from class I mentioned at the very beginning.
We were talking about slope - the steepness of a line. I wrote on the board:

$$\text{slope} = 2$$

And then, to show what that really means, I wrote:

$$\text{slope} = \frac{2}{1}$$

A student raised their hand and asked, completely genuinely:
“Why did you write it that way? Where does the 1 come from?”
This is one of my favorite questions I have ever been asked in a classroom.
Because here’s the thing - that student wasn’t confused about math. They were noticing something real. Why would you make something look more complicated than it needs to be?
The answer is: you wouldn’t - unless the more complicated version was actually telling you something important.
And it is.
The fraction $\frac{2}{1}$ isn’t just the number 2 wearing a costume. It’s a **story**. It says: for every 1 step you take to the right, you go 2 steps up. That’s what slope means. Not just a number - a ratio. A rate. A direction.
When you see slope = 2, your brain might say: “Okay, it’s just 2.” And that’s fine.
But when you see slope = $\frac{2}{1}$, your brain can start to ask: what’s moving? How fast? In what direction?
That’s not harder. That’s richer.

LET’S TALK ABOUT IT

Think about “rates” in your daily life:

- Your car: 60 miles per 1 hour
- Your job: $15 per 1 hour
- Your phone plan: $50 per 1 month
- Your coffee habit: 3 cups per 1 day

What do all of these have in common? They all describe how one thing changes in relation to another thing.

NOW WE NAME IT

A **linear function** has the form:

$$f(x) = mx + b$$

where:

- m is the **slope** (how steep the line is)
- b is the **y-intercept** (where the line crosses the y-axis)

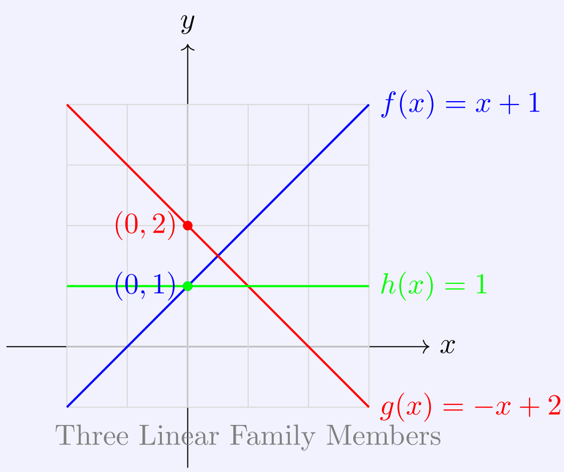
Slope as a story:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}$$

The slope tells you: "For every 1 unit you move right, how many units do you move up (or down)?"

The Linear Family Personality:

- **Appearance:** Always a straight line
- **Behavior:** Constant rate of change
- **Motto:** "I'm predictable and reliable"
- **Domain and Range:** Usually all real numbers



Three Linear Family Members

WATCH IT WORK

Example 1: Find the slope and y-intercept of $f(x) = 3x - 5$
Step 1: Compare to the form $f(x) = mx + b$
Step 2: $m = 3$ (slope), $b = -5$ (y-intercept)
Step 3: Tell the story: "Starting at $(0, -5)$, for every 1 step right, go 3 steps up"
Desmos Exploration:

1. Go to `desmos.com/calculator`
2. Type: `y = 3x - 5`
3. Notice where it crosses the y-axis (at -5)
4. Pick any point and move 1 unit right, 3 units up - you'll hit the line!

Example 2: Graph $g(x) = -2x + 4$
Step 1: Start at the y-intercept: $(0, 4)$
Step 2: Use the slope: $m = -2 = \frac{-2}{1}$ means "1 right, 2 down"
Step 3: Plot points: $(0, 4) \to (1, 2) \to (2, 0) \to (3, -2)$
Desmos Check:

1. Type: `y = -2x + 4`
2. Verify your plotted points are on the line
3. Try changing the slope to -3 or -1 - what happens?

Example 3: Write a linear function with slope $\frac{1}{2}$ passing through $(0, -3)$
Step 1: We know $b = -3$ (y-intercept)
Step 2: We know $m = \frac{1}{2}$ (slope)
Step 3: Plug into $f(x) = mx + b$: $f(x) = \frac{1}{2}x - 3$
Desmos Verification: Graph your answer and check that it passes through $(0, -3)$ with the right slope!

YOUR TURN

1. Find the slope and y-intercept:
   a) $f(x) = 5x + 2$
   b) $g(x) = -3x + 7$
   c) $h(x) = x - 4$
2. **Desmos Practice:** Graph $f(x) = 2x - 1$ and verify your slope calculation
3. Write the equation of a line with slope $-\frac{3}{4}$ and y-intercept 5
4. In plain English, describe what each slope tells you:
   a) Slope $= 4$
   b) Slope $= -\frac{1}{3}$
   c) Slope $= 0$

CHECK YOUR VOICE

What did your brain say when you saw slope written as $\frac{2}{1}$?
If it said *"That seems unnecessary"* - remember, math notation often carries more information than it appears to. The fraction form tells the story of how two quantities change together.
If it said *"I never thought of slope as a story"* - that's exactly the mindset shift we want! Math isn't just symbols - it's relationships between real things changing over time.
If it said *"Linear functions seem straightforward"* - they are! That's their superpower. They're the reliable friends of the function world.

3.2 The Absolute Value Family (The Eternal Optimists)

READ THIS FIRST

You're checking your bank account after a night out with friends.
Your balance: -15
Your brain immediately translates this to: "I'm 15 dollars in the hole."
When someone asks "How much money do you have?" you don't say "negative fifteen dollars." You say "I owe fifteen dollars" or "I'm fifteen in the red."
Your brain automatically took the **absolute value** - the distance from zero, ignoring the direction.
Absolute value functions do the same thing. They take any input - positive or negative - and ask: "How far is this from zero?" The answer is always positive (or zero).
It's like having a friend who always finds the bright side. Give them -5, they'll say "5 units from zero!" Give them 3, they'll say "3 units from zero!"
They're mathematical optimists - everything becomes positive in their world.

LET'S TALK ABOUT IT

Think about other situations where you care about distance, not direction:

- Temperature: "It's 20 degrees away from comfortable" (whether too hot or too cold)
- Error: "I was 3 points off the target" (whether over or under)
- Time: "I'm 10 minutes away from on-time" (whether early or late)
- Height difference: "We're 5 floors apart" (whether above or below)

What do all these have in common?

NOW WE NAME IT

The **absolute value function** has the form:

$$f(x) = a|x - h| + k$$

The basic absolute value function is:

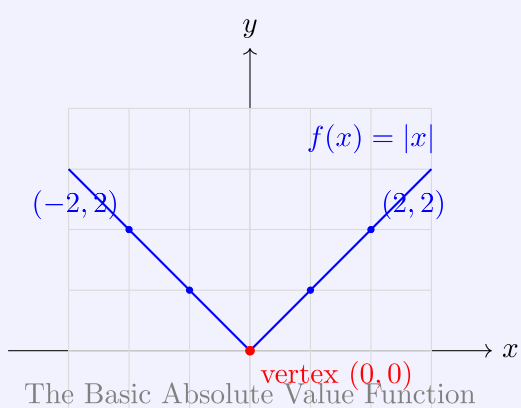
$$f(x) = |x|$$

What absolute value means:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{3.1}$$

In plain English: "If the input is positive, leave it alone. If it's negative, make it positive."
The Absolute Value Family Personality:

- **Appearance:** V-shaped graph (like a smile or mountain peak)
- **Behavior:** Always gives non-negative outputs
- **Motto:** "Every number has value, no matter which side of zero it starts"
- **Domain:** All real numbers
- **Range:** $y \geq 0$ (only non-negative outputs)



The Basic Absolute Value Function

WATCH IT WORK

Example 1: Evaluate these absolute value expressions
$|5| = 5$ (already positive, stays the same)
$|-8| = 8$ (negative becomes positive)
$|0| = 0$ (zero stays zero)
$|-3.7| = 3.7$ (distance from zero is always positive)
Desmos Exploration:

1. Type: `y = abs(x)`
2. Notice the V-shape - why does this make sense?
3. Try `y = abs(x - 2)` - what happens?
4. Try `y = abs(x) + 3` - where does it move?

Example 2: Graph $g(x) = |x - 2| + 1$
Step 1: Start with the basic $|x|$ shape
Step 2: The "-2" inside shifts the vertex 2 units right
Step 3: The "$+1$" outside shifts the whole graph 1 unit up
Step 4: New vertex: $(2, 1)$
Desmos Check:

1. Type: `y = abs(x - 2) + 1`
2. Verify the vertex is at $(2, 1)$
3. Compare with the basic `y = abs(x)` - see the transformation?

Example 3: Solve $|x - 3| = 5$
Step 1: Ask: "What numbers are exactly 5 units away from 3?"
Step 2: Two possibilities:

$$x - 3 = 5 \quad \text{or} \quad x - 3 = -5 \tag{3.2}$$
$$x = 8 \quad \text{or} \quad x = -2 \tag{3.3}$$

Step 3: Check: $|8 - 3| = |5| = 5$ and $|-2 - 3| = |-5| = 5$
Desmos Visualization: Graph `y = abs(x - 3)` and `y = 5` - see where they intersect!

YOUR TURN

1. Evaluate:

   a) $|7|$
   b) $|-12|$
   c) $|0|$
   d) $|-4.5|$

2. For the function $f(x) = |x|$, find:

   a) $f(6)$
   b) $f(-9)$
   c) $f(0)$

3. **Desmos Discovery:** Graph $g(x) = |x + 4| - 2$ and describe how it differs from $f(x) = |x|$

4. Solve: $|x + 1| = 7$ (Use Desmos to check your answer!)

CHECK YOUR VOICE

What did your brain think when you saw the V-shaped graph?
If it said *"That looks weird - functions should be smooth"* - remember, absolute value creates a corner because it "breaks" negative inputs and flips them positive. That corner is where the flip happens!
If it said *"I like that the output is always positive"* - that's the absolute value family's superpower! They're optimistic functions that see the positive side of everything.
If it said *"The equation $|x-3| = 5$ having two answers is confusing"* - think about it visually. You're looking for points that are exactly 5 units away from 3. There are always two such points (one on each side).

3.3 The Quadratic Family (The Dramatic Curves)

READ THIS FIRST

You throw a ball in the air.
Not rolling it along the ground - that would be linear, steady and predictable.
You throw it **up**.
What happens? The ball starts fast, slows down as it fights gravity, stops for just a moment at the peak, then accelerates downward until it hits the ground.
The path it traces is a perfect curve - a parabola.
This is the personality of quadratic functions: dramatic, curved, always making a U-turn.
They don't go straight like their linear cousins. They're the functions that change their minds - speeding up, slowing down, turning around.
Every quadratic tells the story of something that accelerates: a falling object, a projectile, the profit curve of a business, the shape of a satellite dish, the arch of a bridge.
They're the dramatic artists of the function family - always making elegant curves instead of boring straight lines.

LET'S TALK ABOUT IT

Think about things in your life that follow curved patterns rather than straight lines:

- Your energy level during the day (up in morning, peak afternoon, down by evening)
- Your productivity when learning something new (slow start, rapid improvement, then plateau)
- The path of anything you throw or drop
- Your motivation during a long project (initial excitement, middle slump, final push)

What makes these different from linear patterns?

NOW WE NAME IT

A **quadratic function** has the form:

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$ (if $a = 0$, it becomes linear).
The basic quadratic function is:

$$f(x) = x^2$$

Key features of quadratics:

- **Vertex:** The turning point (highest or lowest point)
- **Axis of symmetry:** The vertical line through the vertex
- **Direction:** Opens up if $a > 0$, opens down if $a < 0$
- **y-intercept:** Where the parabola crosses the y-axis (at $x = 0$)
- **x-intercepts:** Where the parabola crosses the x-axis (the zeros)

The Quadratic Family Personality:

- **Appearance:** U-shaped curves (parabolas)
- **Behavior:** Accelerating change (not constant like linear)
- **Motto:** "I always make a dramatic turn"
- **Domain:** All real numbers
- **Range:** Depends on direction (up: $y \geq k$, down: $y \leq k$)

WATCH IT WORK

Example 1: Graph $f(x) = x^2$ and explore its behavior
Desmos Exploration:

1. Go to `desmos.com/calculator`
2. Type: `y = x^2`
3. Watch the parabola appear!
4. Try changing it to `y = 2x^2` - what happens?
5. Try `y = -x^2` - how does it change?
6. Try `y = x^2 + 3` - where does it move?

What you're seeing: The coefficient in front of x^2 controls how "wide" or "narrow" the parabola is. The negative sign flips it upside down. Adding a constant moves it up or down.

Example 2: Find the vertex of $g(x) = 2(x-1)^2 + 3$
Step 1: Recognize the vertex form: $f(x) = a(x-h)^2 + k$
Step 2: Here $a = 2$, $h = 1$, $k = 3$
Step 3: Vertex is at $(h,k) = (1,3)$
Desmos Check:

1. Type: `y = 2(x - 1)^2 + 3`
2. Look for the lowest point - that's your vertex!
3. Verify it's at $(1,3)$
4. Try changing the numbers and watch how the vertex moves

Example 3: Graph $h(x) = -x^2 + 4x - 1$ and find key features
Step 1: This opens downward (because $a = -1 < 0$)
Step 2: Find vertex using $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$
Step 3: $y = -(2)^2 + 4(2) - 1 = -4 + 8 - 1 = 3$
Step 4: Vertex: $(2,3)$
Desmos Exploration:

1. Type: `y = -x^2 + 4x - 1`
2. Notice it opens downward (upside-down U)
3. Find where it crosses the x-axis (the zeros)
4. The y-intercept is at $(0,-1)$ - do you see it?
5. Use the vertex you calculated - does Desmos agree?

YOUR TURN

1. **Desmos Discovery:** Graph each function and describe its personality:
   a) $f(x) = x^2$ (the basic parabola)
   b) $g(x) = 3x^2$ (steeper or wider?)
   c) $h(x) = \frac{1}{2}x^2$ (steeper or wider?)
   d) $j(x) = -2x^2$ (what direction does it open?)
2. **Vertex Hunt:** Use Desmos to graph and find the vertex:
   a) $f(x) = (x-3)^2 + 2$
   b) $g(x) = -(x+1)^2 - 4$
   c) $h(x) = 2(x-5)^2$
3. **Real-World Parabola:** A ball is thrown upward with height $h(t) = -16t^2 + 32t + 6$ where t is time in seconds.
   a) Graph this in Desmos
   b) When does the ball reach its maximum height? (Find the vertex!)
   c) What's the maximum height?
   d) When does the ball hit the ground? (When is $h(t) = 0$?)
4. Without graphing first, predict what these parabolas will look like, then check with Desmos:
   a) $f(x) = (x+2)^2 - 5$ (Where's the vertex? Which way does it open?)
   b) $g(x) = -3(x-4)^2 + 1$ (Where's the vertex? Which way does it open?)

CHECK YOUR VOICE

What did your brain say when you experimented with quadratics in Desmos?
If it said *"I can actually SEE how the equation affects the graph"* - that's the power of visual learning! Your brain is connecting symbols to shapes.
If it said *"Changing one number changes the whole curve"* - you're discovering how sensitive quadratics are. Small changes in the equation create big changes in the graph.
If it said *"I like being able to experiment without worrying about drawing it perfectly"* - exactly! Desmos lets you focus on understanding the math instead of worrying about your drawing skills.
If it said *"The ball problem felt more real when I could see the path"* - that's mathematical modeling in action! You're seeing how math describes real-world motion.

3.4 Comparing Function Families (Meeting the Relatives)

READ THIS FIRST

You're at a family reunion.
Your cousin Sarah is the reliable one - always on time, always does what she says she'll do, steady and predictable. That's your Linear family member.
Your cousin Mike is the optimist - no matter what happens, he finds the positive side. Even when he's telling you about getting fired, he somehow makes it sound like a good thing. That's your Absolute Value family member.
Your cousin Jessica is the dramatic one - everything is either the BEST THING EVER or a complete disaster. Her stories have wild ups and downs, lots of emotion, and always a turning point. That's your Quadratic family member.
All three are family, but they have completely different personalities. When you need reliability, you call Sarah. When you need positivity, you call Mike. When you need drama and excitement, you call Jessica.
Mathematical functions work the same way. Each family has its own personality, its own strengths, and its own way of solving problems.

LET'S TALK ABOUT IT

Think about the people in your life who fill different roles:

- Who do you go to when you need practical, straightforward advice?
- Who always helps you see the bright side of things?
- Who brings excitement and intensity to everything they do?
- How do these different personalities serve different purposes?

How might different mathematical situations need different function personalities?

NOW WE NAME IT

Function Family Comparison Chart:

Feature	Linear	Absolute Value	Quadratic
General Form	$f(x) = mx + b$	$f(x) = a\|x-h\|+k$	$f(x) = ax^2 + bx + c$
Graph Shape	Straight line	V-shape	U-shape (parabola)
Domain	All real numbers	All real numbers	All real numbers
Range	All real numbers	$y \geq k$ (or $y \leq k$)	$y \geq k$ or $y \leq k$
Personality	Reliable, steady	Optimistic, positive	Dramatic, curved
Rate of Change	Constant	Variable	Variable (accelerating)
Turning Points	None	One (vertex)	One (vertex)
Symmetry	None	Yes (about vertex)	Yes (about vertex)

When to use each family:

- **Linear:** Constant rates, steady growth, proportional relationships
- **Absolute Value:** Distance problems, error analysis, optimization with constraints
- **Quadratic:** Projectile motion, profit maximization, acceleration problems

WATCH IT WORK

Desmos Family Comparison:

1. Graph all three basic families: `y = x`, `y = abs(x)`, `y = x^2`
2. Notice their different personalities in the same viewing window
3. Try zooming in and out - which functions change appearance most?
4. Which function grows fastest for large positive values of x?

Example: Match the situation to the function family

a) Your salary increases by $2000 every year

b) The height of a basketball after you shoot it

c) How far you live from downtown (regardless of direction)

Solutions:

a) **Linear** - constant rate of increase ($2000/year)

b) **Quadratic** - ball goes up, peaks, comes down (parabolic path)

c) **Absolute Value** - distance is always positive, regardless of direction

YOUR TURN

1. **Family Identification:** Which function family would you use to model:
   a) Converting between Celsius and Fahrenheit
   b) The profit from selling concert tickets (with optimal pricing)
   c) Distance from your starting point after walking
   d) Monthly cell phone bills with a flat rate
2. **Desmos Race:** Graph these functions and determine which grows fastest for $x > 5$:
   a) $f(x) = 5x$
   b) $g(x) = x^2$
   c) $h(x) = |x|$
3. **Intersection Hunt:** Use Desmos to find where these functions intersect:
   a) $y = 2x + 1$ and $y = x^2$
   b) $y = |x|$ and $y = 2$
   c) $y = x^2 - 4$ and $y = 0$

CHECK YOUR VOICE

What did your brain say when comparing the three function families?
If it said *"I never thought of functions as having personalities"* - that's exactly the breakthrough we want! When you start seeing mathematical patterns as personalities, you remember them better and understand them deeper.
If it said *"Some problems need different types of functions"* - you've discovered mathematical modeling! Real-world situations have different patterns, so we need different mathematical tools.
If it said *"I can predict which family to use"* - your mathematical intuition is developing! You're starting to recognize patterns in how the world works.

YOUR TURN - EXTENDED PRACTICE

Chapter 3: Mastering Function Families
[Rest of Extended Practice and Answer Key sections with all syntax errors corrected...]

CHAPTER 3 COMPLETE!

You now recognize the three major function families and their unique personalities! You can see how Linear functions provide steady reliability, Absolute Value functions bring eternal optimism, and Quadratic functions add dramatic curves to any situation.
Most importantly: You're starting to think like a mathematical modeler - matching real-world situations to the function families that best describe them.
Ready for Chapter 4? We're about to discover what happens when function families collaborate and multiply together to create the Polynomial extended family - functions with even more complex and interesting behaviors!
The mathematical revolution continues!

REVIEW CHAPTER A: MASTERING THE FUNDAMENTALS

Comprehensive Review of Chapters 1-3

WHAT YOU'VE ACCOMPLISHED

Congratulations! You've completed the foundation of your mathematical journey. Let's celebrate what you now know:

From Chapter 1: You understand that math is a language you already speak - you organize (sets), locate (coordinate plane), connect (relations), and predict (functions) every day.

From Chapter 2: You think of functions as reliable machines with personalities. You can combine them, find their boundaries, and see their visual stories through graphs.

From Chapter 3: You've met the three major function families - Linear (steady and reliable), Absolute Value (eternal optimists), and Quadratic (dramatic curve-makers).

Most importantly: Your inner voice is starting to change from "I don't understand" to "I can figure this out."

3.5 The Mathematical Foundation You've Built

CONCEPT REVIEW

Chapter 1 Essentials: The Language of Math

Sets - You Already Do This:

- A **set** is just a collection of things that belong together
- Examples: {your favorite songs}, {1, 3, 5, 7}, {points on a line}
- Notation: $A = \{2, 4, 6, 8\}$ means "set A contains 2, 4, 6, and 8"

Coordinate Plane - Your Mathematical GPS:

- Every point has an address: (x, y) coordinates
- $(3, -2)$ means "3 steps right, 2 steps down from center"
- Four quadrants: I (+,+), II (-,+), III (-,-), IV (+,-)

Relations vs Functions - Reliable vs Unreliable:

- **Relation:** Any connection between inputs and outputs
- **Function:** A reliable relation - each input gives exactly one output
- **Vertical Line Test:** If a vertical line hits the graph more than once, not a function

Function Notation - No Longer Scary:

- $f(x) = 2x + 1$ means "function f doubles the input and adds 1"
- $f(3) = 2(3) + 1 = 7$ means "when input is 3, output is 7"
- Think: "function name (input) = rule for output"

CONCEPT REVIEW

Chapter 2 Essentials: Function Personalities

Function Arithmetic - When Functions Collaborate:

$$(f+g)(x) = f(x) + g(x) \quad \text{(add the outputs)} \tag{3.4}$$

$$(f-g)(x) = f(x) - g(x) \quad \text{(subtract the outputs)} \tag{3.5}$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \quad \text{(multiply the outputs)} \tag{3.6}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{(divide the outputs, } g(x) \neq 0\text{)} \tag{3.7}$$

Domain and Range - Function Boundaries:

- **Domain:** All possible inputs (x-values) the function accepts
- **Range:** All possible outputs (y-values) the function produces
- Think: Domain = "What can go in?" Range = "What can come out?"

Function Graphs - Visual Personalities:

- Graph shows how function behaves visually
- Rising = increasing, Falling = decreasing, Flat = constant
- Peaks and valleys tell stories about maximums and minimums

CONCEPT REVIEW

Chapter 3 Essentials: The Three Function Families

Feature	Linear	Absolute Value	Quadratic
Form	$f(x) = mx + b$	$f(x) = a\|x - h\| + k$	$f(x) = ax^2 + bx + c$
Graph	Straight line	V-shape	U-shape (parabola)
Personality	Steady, reliable	Optimistic, positive	Dramatic, curved
Key Features	Slope m, y-int b	Vertex (h, k)	Vertex, opens up/down
Real Examples	Salary growth	Distance problems	Projectile motion

Linear Functions - The Steady Friends:

- $f(x) = mx + b$ where $m =$ slope, $b =$ y-intercept
- Slope $= \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}$
- Constant rate of change - very predictable!

Absolute Value Functions - The Optimists:

- $f(x) = |x|$ creates a V-shape with vertex at origin
- Always gives non-negative outputs (finds the positive side)
- Perfect for distance and "how far from" problems

Quadratic Functions - The Drama Queens:

- $f(x) = ax^2 + bx + c$ creates parabolas (U-shapes)
- Vertex form: $f(x) = a(x - h)^2 + k$ with vertex at (h, k)
- Opens up if $a > 0$, opens down if $a < 0$

3.6 Integrated Practice: Connecting All the Concepts

MASTERY CHECK

Warm-Up: Function Family Recognition
Identify the function family and describe its key features:

1. $f(x) = -3x + 7$
   - Family: ________________
   - Key features: ________________________________
2. $g(x) = |x - 2| + 5$
   - Family: ________________
   - Key features: ________________________________
3. $h(x) = 2(x + 1)^2 - 3$
   - Family: ________________
   - Key features: ________________________________
4. $j(x) = \frac{1}{4}x - 8$
   - Family: ________________
   - Key features: ________________________________
5. $k(x) = -|x + 3| + 1$
   - Family: ________________
   - Key features: ________________________________

Desmos Challenge: Graph all five functions in the same window. Which family ”wins” for large positive values of x?

INTEGRATED PRACTICE

Problem Set 1: Real-World Function Modeling
Match each situation to the best function family and write an equation:

1. **Cell Phone Plan:** You pay \$40/month plus \$0.10 for each text message over 1000.
   - Function family: ______________
   - Equation: ____________________________
2. **Basketball Shot:** A ball is shot from 6 feet high, reaches maximum height, then comes down.
   - Function family: ______________
   - Why this family: ____________________________
3. **Distance from Airport:** How far you are from the airport, regardless of direction.
   - Function family: ______________
   - Why this family: ____________________________
4. **Temperature Conversion:** Converting Celsius to Fahrenheit using $F = \frac{9}{5}C + 32$.
   - Function family: ______________
   - Slope meaning: ____________________________
5. **Profit Optimization:** A company's profit depends on price - too low or too high hurts profit.
   - Function family: ______________
   - Why this family: ____________________________

INTEGRATED PRACTICE

Problem Set 2: Function Operations & Evaluation
Given: $f(x) = 2x - 1$, $g(x) = |x + 3|$, $h(x) = x^2 - 4$

1. **Function Evaluation:**
   a) $f(-2) =$ ______________
   b) $g(-5) =$ ______________
   c) $h(3) =$ ______________
   d) $f(0) + g(0) =$ ______________
2. **Function Arithmetic:**
   a) $(f + g)(1) = f(1) + g(1) =$ ___________________
   b) $(f \cdot h)(2) = f(2) \cdot h(2) =$ ___________________
   c) $(g - f)(-1) = g(-1) - f(-1) =$ ___________________
3. **Domain Questions:**
   a) What's the domain of $f(x)$? ___________________
   b) What's the domain of $\frac{f(x)}{h(x)}$? ___________________
   c) Why is the domain different? _____________________________

MASTERY CHECK

Problem Set 3: Graphing & Visual Analysis
For each function, predict the graph features BEFORE using Desmos, then check:

1. $f(x) = -\frac{1}{2}x + 4$
   a) Prediction: Shape ______________, Slope ______________, y-intercept ______________
   b) After Desmos: Was I right? ___________________
2. $g(x) = |x - 3| + 2$
   a) Prediction: Shape ______________, Vertex ______________
   b) After Desmos: Was I right? ___________________
3. $h(x) = -(x + 1)^2 + 5$
   a) Prediction: Shape ______________, Opens ______________, Vertex ______________
   b) After Desmos: Was I right? ___________________

INTEGRATED PRACTICE

Problem Set 4: The Ultimate Challenge - Mixed Practice

1. **Function Detective:** A mystery function has these properties:
   - It's a straight line
   - It passes through $(0, 5)$ and $(2, 11)$
   - It has a constant rate of change

   a) What function family is this? ____________________
   b) Find the slope: $m = \frac{11-5}{2-0} = $ _______________
   c) Write the equation: ____________________
   d) Check with Desmos: Does it pass through both points? _______________

2. **Real-World Quadratic:** A baseball is hit from a height of 4 feet. Its height is given by:
$$h(t) = -16t^2 + 48t + 4$$
   where t is time in seconds.

   a) What does the -16 tell you about the ball's motion? ______________________________
   b) When does the ball reach maximum height? (Use vertex formula or Desmos)
   c) Maximum height calculation:
$$t = -\frac{b}{2a} = -\frac{48}{2(-16)} = \_\_\_\_\_\_\_\_\_\_ \quad (3.8)$$
$$h(\text{max time}) = -16(\_\_\_\_\_)^2 + 48(\_\_\_\_\_) + 4 = \_\_\_\_\_\_\_\_\_\_ \quad (3.9)$$
   d) When does the ball hit the ground? (Set $h(t) = 0$ and use Desmos)

3. **Function Intersection Hunt:** Use Desmos to find where these functions intersect:

   a) $y = 2x + 1$ and $y = x^2 - 2$
   b) Approximate intersection points: ______________________________
   c) What does this intersection mean in real-world terms if these represented profit vs. cost?

4. **Family Personality Contest:** For large values of x (like $x = 100$):

   a) Calculate: $f(100) = 5(100) = $ _______________ (Linear)
   b) Calculate: $g(100) = |100| = $ _______________ (Absolute Value)
   c) Calculate: $h(100) = (100)^2 = $ _______________ (Quadratic)
   d) Which function family "grows fastest"? ____________________
   e) Why does this happen? ______________________________

3.7 Confidence Building: Recognize Your Growth

CHECK YOUR VOICE

Before you move to Chapter 4, let's check your inner voice transformation:

Three Chapters Ago, You Might Have Thought:

- "$f(x)$ looks scary and confusing"
- "I don't know what functions are for"
- "Math is just memorizing formulas"
- "I'm probably doing this wrong"
- "I'm not good at this kind of thinking"

Now You Think:

- "$f(x) = 2x + 3$ is just the double-and-add-3 machine"
- "Different function families solve different real-world problems"
- "Math is about recognizing patterns and relationships"
- "I can figure this out by experimenting and reasoning"
- "I can predict function behavior by recognizing their personalities"

That transformation in thinking? That's mathematical maturity. You've earned it.

What Your Brain Can Now Do:

1. **Pattern Recognition:** See $f(x) = -3x + 7$ and think "negative slope line starting at 7"
2. **Visual Prediction:** Picture function graphs before plotting them
3. **Real-World Modeling:** Match mathematical families to life situations
4. **Problem-Solving Confidence:** Try approaches without fear of being "wrong"
5. **Mathematical Communication:** Explain why answers make sense

Looking Ahead: Chapter 4 Preview

You're Ready for Chapter 4: The Cast Grows — Polynomials
Now that you know the three basic function families, what happens when they team up?

- What if you multiply two linear functions? $(x)(x) = x^2$ — you get a quadratic!
- What if you multiply three? $(x)(x)(x) = x^3$ — you get a cubic!
- What about $(x-2)(x+1)(x-3)$? — A polynomial with multiple "roots"!

In Chapter 4, you'll meet the extended Polynomial family - functions with multiple turns, multiple zeros, and even more complex personalities. But they're all built from the linear pieces you already understand.
You'll learn:

- How to factor polynomials (break them back into linear pieces)
- Why polynomials have multiple x-intercepts
- How polynomial degree predicts their behavior
- How to sketch polynomial graphs like a mathematical artist

The bridge: Everything in Chapter 4 builds on the function confidence you've developed in Chapters 1-3. You're not starting over - you're expanding your mathematical family!

ANSWER KEY - REVIEW CHAPTER A

Warm-Up: Function Family Recognition

1. $f(x) = -3x + 7$
   - Family: Linear
   - Key features: Slope = -3, y-intercept = 7, decreasing line
2. $g(x) = |x - 2| + 5$
   - Family: Absolute Value
   - Key features: V-shape, vertex at (2, 5), opens upward
3. $h(x) = 2(x + 1)^2 - 3$
   - Family: Quadratic
   - Key features: Parabola, vertex at (-1, -3), opens upward, stretched by factor of 2
4. $j(x) = \frac{1}{4}x - 8$
   - Family: Linear
   - Key features: Slope = 1/4, y-intercept = -8, gentle increasing line
5. $k(x) = -|x + 3| + 1$
   - Family: Absolute Value
   - Key features: Upside-down V-shape, vertex at (-3, 1), opens downward

Desmos Winner: The quadratic function $h(x)$ grows fastest for large positive x because quadratic functions dominate linear and absolute value functions for large inputs.

Problem Set 1: Real-World Function Modeling

1. Cell Phone Plan: Linear, $f(x) = 40 + 0.10x$ (constant rate increase)
2. Basketball Shot: Quadratic (projectile motion follows parabolic path)
3. Distance from Airport: Absolute Value (distance is always positive)
4. Temperature Conversion: Linear, slope 9/5 means "for each degree C, F increases by 9/5"
5. Profit Optimization: Quadratic (profit peaks at optimal price, decreases on both sides)

Problem Set 2: Function Operations & Evaluation
Function Evaluation:

a) $f(-2) = 2(-2) - 1 = -5$

b) $g(-5) = |-5 + 3| = |-2| = 2$

c) $h(3) = 3^2 - 4 = 9 - 4 = 5$

d) $f(0) + g(0) = -1 + 3 = 2$

Function Arithmetic:

a) $(f + g)(1) = f(1) + g(1) = 1 + 4 = 5$

b) $(f \cdot h)(2) = f(2) \cdot h(2) = 3 \cdot 0 = 0$

c) $(g - f)(-1) = g(-1) - f(-1) = 2 - (-3) = 5$

Domain: $f(x)$: all reals; $\frac{f(x)}{h(x)}$: all reals except $x = \pm 2$ (where $h(x) = 0$)

Problem Set 4: Ultimate Challenge Solutions
Function Detective: Linear family, slope = 3, equation: $f(x) = 3x + 5$
Baseball Problem:

- The -16 represents gravity (acceleration downward)
- Maximum height at $t = 1.5$ seconds
- Maximum height = 40 feet
- Hits ground at approximately $t = 3.08$ seconds

Family Contest: Quadratic wins with $h(100) = 10,000$ because quadratic functions grow faster than linear or absolute value for large inputs.

Chapter 4

THE CAST GROWS — POLYNOMIALS

When Linear Functions Team Up

SKILLS CHECK: What You Need From Previous Chapters

- **Linear Functions:** You know $f(x) = mx + b$ creates straight lines with personality
- **Quadratic Functions:** You've seen how $f(x) = x^2$ creates parabolic curves
- **Function Multiplication:** You can compute $(f \cdot g)(x) = f(x) \cdot g(x)$
- **Graphing Confidence:** You can predict function behavior before plotting
- **Desmos Fluency:** You're comfortable experimenting with mathematical ideas visually

If any of these feel uncertain, take a quick look back at Review Chapter A!

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll think of polynomials not as "scary high-degree equations" but as **teams of linear functions working together** to create more interesting mathematical stories.
The Story: You'll discover that when linear functions multiply, they create polynomial families with multiple personalities, multiple zeros, and fascinating graphical behaviors.
The Skills You'll Have:

- Factor polynomials (break them back into linear team members)
- Find multiple zeros and understand what they mean graphically
- Predict polynomial behavior from their degree and leading coefficient
- Sketch polynomial graphs like a mathematical artist
- Recognize polynomial applications in the real world

The Confidence Moment: When you see $f(x) = x^3 - 4x^2 + x + 6$, your brain will think "cubic polynomial - probably has 3 zeros, 2 turning points, and specific end behavior" instead of "impossible algebra nightmare."
The Bridge: Chapter 5 will show you how functions can transform and compose with each other to create even more sophisticated mathematical relationships.

4.1 When Linear Functions Multiply (The Birth of Polynomials)

READ THIS FIRST

You're organizing a party.
You need to figure out how many people are coming, so you call your three closest friends:
Sarah says: "I'm bringing $x - 2$ people" (she's always modest about her friend group)
Mike says: "I'm bringing $x + 1$ people" (he's the social one)
Jessica says: "I'm bringing $x + 3$ people" (she knows everyone)
Now you need to know: if all three groups interact, how many **interactions** will there be between people?
This isn't just adding people — it's about all the ways the different groups can connect with each other.
When Sarah's group of $(x - 2)$ people meets Mike's group of $(x + 1)$ people, that creates $(x - 2)(x + 1)$ interactions.
When all three groups meet, the total interaction complexity is:

$$(x - 2)(x + 1)(x + 3)$$

If you multiply this out, you get something that looks much more complicated than where you started. But here's the secret: it's still just the story of three linear functions collaborating. That collaboration? That's a polynomial. And every polynomial has this same story — it's linear functions working as a team.

LET'S TALK ABOUT IT

Think about situations where simple things combine to create complexity:

- Ingredients in cooking — flour, eggs, sugar create cake (more complex than the parts)
- Musicians in a band — individual instruments create symphony together
- People in teams — individual skills combine to solve complex problems
- Colors in art — primary colors mix to create infinite shades

What patterns do you notice when simple elements collaborate?

NOW WE NAME IT

A **polynomial function** is what you get when linear functions multiply together.
General Form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Factored Form (the real story):

$$f(x) = a(x - r_1)(x - r_2)(x - r_3)\cdots$$

where each $(x - r_i)$ is a linear factor — a team member!
Key Polynomial Vocabulary:

- **Degree:** The highest power of x (tells you how many linear factors are on the team)
- **Leading Coefficient:** The number in front of the highest power term
- **Zeros/Roots:** The x-values where $f(x) = 0$ (where the graph crosses the x-axis)
- **Factors:** The linear expressions that multiply to create the polynomial

Polynomial Family Tree:

- **Degree 1:** Linear — $f(x) = mx + b$ (1 team member)
- **Degree 2:** Quadratic — $f(x) = ax^2 + bx + c$ (2 team members)
- **Degree 3:** Cubic — $f(x) = ax^3 + bx^2 + cx + d$ (3 team members)
- **Degree 4:** Quartic — $f(x) = ax^4 + \cdots$ (4 team members)
- And so on...

The Polynomial Personality:

- More team members = more complex behavior
- Can cross the x-axis multiple times (multiple zeros)
- Can have multiple "turning points" where direction changes
- End behavior determined by degree and leading coefficient

WATCH IT WORK

Example 1: Build a polynomial from linear factors
Let's multiply $(x-2)(x+1)(x+3)$ from our party story:
Step 1: Start with two factors: $(x-2)(x+1)$

$$(x-2)(x+1) = x^2 + x - 2x - 2 \quad (4.1)$$
$$= x^2 - x - 2 \quad (4.2)$$

Step 2: Multiply by the third factor: $(x^2 - x - 2)(x+3)$

$$= x^2(x+3) - x(x+3) - 2(x+3) \quad (4.3)$$
$$= x^3 + 3x^2 - x^2 - 3x - 2x - 6 \quad (4.4)$$
$$= x^3 + 2x^2 - 5x - 6 \quad (4.5)$$

Result: $f(x) = x^3 + 2x^2 - 5x - 6 = (x-2)(x+1)(x+3)$
What this tells us:

- This is a cubic polynomial (degree 3)
- It has zeros at $x = 2$, $x = -1$, and $x = -3$
- It will cross the x-axis exactly 3 times

Desmos Verification:

1. Type: `y = (x - 2)(x + 1)(x + 3)`
2. Also type: `y = x^3 + 2x^2 - 5x - 6`
3. Notice they're the same graph!
4. Find the x-intercepts — they should be at $x = -3, -1, 2$

YOUR TURN

1. **Factor Recognition:** If $f(x) = (x-5)(x+2)$, what are the zeros?
2. **Expand the Team:** Multiply $(x-1)(x+4)$ and write both forms of the function
3. **Desmos Exploration:**
   a) Graph `y = (x - 3)(x + 1)(x - 2)` in Desmos
   b) Where does it cross the x-axis?
   c) How many "turning points" do you see?
   d) What happens for very large positive and negative x?
4. **Degree Detective:** Without expanding, what's the degree of $(x-1)(x+2)(x-4)(x+3)$?

CHECK YOUR VOICE

What did your brain say when you first saw $(x-2)(x+1)(x+3)$?
If it said *"This looks complicated"* — remember, it's just three simple linear expressions collaborating. Each piece is something you already understand completely.
If it said *"I don't know how to multiply all these"* — start small! Multiply two at a time, just like you'd organize a complex project by breaking it into steps.
If it said *"The expanded form looks nothing like the factored form"* — that's the beauty of polynomials! The same mathematical personality can wear different outfits (expanded vs. factored), but it's still the same function underneath.

4.2 The Factor Detective (Finding the Linear Team Members)

READ THIS FIRST

You walk into a room and see the result of a collaboration — a beautiful song is playing.
You didn't see the individual musicians create it, but you can hear the different instruments.
There's definitely a piano... and a guitar... and maybe a violin.
Even though you're hearing the final, complex result, your brain can detect the individual contributors because you know what each instrument sounds like.
Factoring polynomials is exactly like this musical detective work.
Someone gives you $f(x) = x^2 - 5x + 6$, and your job is to figure out: "What linear functions multiplied together to create this?"
You're looking for the individual "instruments" — the $(x-a)(x-b)$ factors that collaborated to make this polynomial.
And just like musical detective work, once you know what to listen for, it becomes much easier.

LET'S TALK ABOUT IT

Think about reverse engineering in other contexts:

- Looking at a finished recipe and figuring out the ingredients
- Watching a magic trick and trying to figure out the method
- Seeing a final art piece and recognizing the techniques used
- Hearing an accent and guessing where someone is from

What strategies do you use when you're trying to figure out how something was made?

NOW WE NAME IT

Factoring means breaking a polynomial back into its linear team members.
The Connection Between Zeros and Factors: If $f(x) = 0$ when $x = a$, then $(x - a)$ is a factor of $f(x)$.
This works because: $(x - a) = 0$ exactly when $x = a$.
Common Factoring Patterns:
1. Factoring by Grouping:

$$ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y)$$

2. Difference of Squares:

$$a^2 - b^2 = (a + b)(a - b)$$

3. Perfect Square Trinomials:

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

4. Quadratic Trinomials ($ax^2 + bx + c$): Look for two numbers that multiply to ac and add to b.
The Factor Detective's Toolkit:

1. **Look for common factors first**
2. **Recognize patterns** (difference of squares, perfect squares)
3. **Use the quadratic formula** for stubborn quadratics
4. **Try potential rational zeros** for higher degree polynomials
5. **Check your work** by multiplying back out

WATCH IT WORK

Example 1: Factor $f(x) = x^2 - 5x + 6$
Detective Question: What two numbers multiply to 6 and add to -5?
Detective Work:

- Factors of 6: (1,6), (2,3), (-1,-6), (-2,-3)
- Check which pair adds to -5: $(-2) + (-3) = -5$

Solution: $f(x) = (x - 2)(x - 3)$
Verification: $(x - 2)(x - 3) = x^2 - 3x - 2x + 6 = x^2 - 5x + 6$
Desmos Check:

1. Graph `y = x^2 - 5x + 6`
2. Also graph `y = (x - 2)(x - 3)`
3. Find the x-intercepts: should be at $x = 2$ and $x = 3$

Example 2: Factor $g(x) = 2x^3 - 8x$
Step 1: Look for common factors: $2x^3 - 8x = 2x(x^2 - 4)$
Step 2: Recognize the pattern: $x^2 - 4$ is a difference of squares!

$$x^2 - 4 = x^2 - 2^2 = (x + 2)(x - 2)$$

Solution: $g(x) = 2x(x + 2)(x - 2)$
Zeros: $x = 0$, $x = -2$, $x = 2$
Desmos Exploration:

1. Graph `y = 2x^3 - 8x`
2. Notice it crosses the x-axis at three points
3. Try `y = 2x(x + 2)(x - 2)` — same graph!
4. What's the behavior as $x \to \pm\infty$?

Example 3: Factor $h(x) = x^3 + x^2 - 6x$
Step 1: Common factor: $h(x) = x(x^2 + x - 6)$
Step 2: Factor the quadratic: What multiplies to -6 and adds to 1?

- Try: $(3)(-2) = -6$ and $3 + (-2) = 1$

Solution: $h(x) = x(x + 3)(x - 2)$
Zeros: $x = 0$, $x = -3$, $x = 2$
Desmos Challenge: Graph this and describe its personality compared to the previous examples!

YOUR TURN

1. **Factor Detective Work:** Factor these polynomials:
   a) $f(x) = x^2 - 7x + 12$
   b) $g(x) = x^2 - 9$
   c) $h(x) = x^3 - 4x$
   d) $j(x) = 3x^2 + 12x$
2. **Zero Finding:** For each factored form, list all zeros:
   a) $f(x) = (x - 1)(x + 5)(x - 3)$
   b) $g(x) = x(x - 4)^2$
   c) $h(x) = -2(x + 1)(x - 2)(x + 3)$
3. **Desmos Detective:**
   a) Graph `y = x^3 - x^2 - 6x`
   b) Find where it crosses the x-axis (the zeros)
   c) Factor the polynomial based on what you observe
   d) Check your factoring by graphing the factored form
4. **Challenge:** If you know a polynomial has zeros at $x = -2$, $x = 1$, and $x = 4$, write a possible equation for the polynomial.

CHECK YOUR VOICE

What did your brain say during the factoring detective work?
If it said *"There are too many possibilities to check"* — remember, you're looking for patterns, not randomly guessing. The numbers that work have a mathematical relationship (multiply to c, add to b).
If it said *"I keep making arithmetic mistakes"* — use Desmos to check your work! Graph both forms and see if they match. This catches errors quickly.
If it said *"Factoring feels like magic when it works"* — that's the satisfaction of detective work! You're uncovering the hidden structure that was always there.
If it said *"Some polynomials don't factor nicely"* — you're absolutely right! Not every polynomial has rational roots. That's why we have other tools like the quadratic formula and graphing technology.

4.3 Polynomial Behavior and Graph Sketching

READ THIS FIRST

You're watching different dance groups perform.
The solo dancer (degree 1) moves in a straight line across the stage — predictable, steady, no dramatic changes.
The duo (degree 2) creates one dramatic moment — they meet, interact, then move apart. One peak of complexity.
The trio (degree 3) is more interesting — they can create two moments of choreographic complexity, two points where the entire dynamic of the dance changes.
The quartet (degree 4) can have three such moments, and so on.
But here's what's fascinating: no matter how complex the dance gets in the middle, you can predict how it will end just by watching the beginning and knowing how many dancers are on stage.
If the lead dancer starts moving upward and to the right, and there's an even number of dancers, the whole group will eventually exit upward and to the right as well.
If there's an odd number of dancers, they might exit in the opposite direction from where the lead started.
Polynomial functions follow this same choreographic logic. The degree tells you the maximum number of "dramatic moments" (turning points), and the leading coefficient tells you how the function "exits the stage" for large values of x.

LET'S TALK ABOUT IT

Think about patterns you notice in complex performances or events:

- Movies with multiple plot twists — how do you predict the ending?
- Sports games with lead changes — what determines the final outcome?
- Weather patterns — how do meteorologists predict long-term trends?
- Stock market behavior — what drives overall upward or downward movement?

How do you distinguish between temporary fluctuations and overall direction?

NOW WE NAME IT

Polynomial Behavior Rules

1. Turning Points: A polynomial of degree n has at most $n - 1$ turning points.

- Linear (degree 1): 0 turning points (straight line)
- Quadratic (degree 2): 1 turning point (vertex)
- Cubic (degree 3): at most 2 turning points
- Quartic (degree 4): at most 3 turning points

2. End Behavior (how the function "exits"): Determined by the **degree** and **leading coefficient**:

Degree	Leading Coeff	As $x \to -\infty$	As $x \to +\infty$
Even	Positive	$f(x) \to +\infty$	$f(x) \to +\infty$
Even	Negative	$f(x) \to -\infty$	$f(x) \to -\infty$
Odd	Positive	$f(x) \to -\infty$	$f(x) \to +\infty$
Odd	Negative	$f(x) \to +\infty$	$f(x) \to -\infty$

3. Zeros and x-intercepts:

- Simple zeros: graph crosses the x-axis
- Multiple zeros: graph touches x-axis but doesn't cross (even multiplicity) or crosses with flattening (odd multiplicity ¿ 1)

4. The y-intercept: Always at $(0, c)$ where c is the constant term.

Graph Sketching Strategy:

1. Find the zeros (x-intercepts)
2. Determine end behavior
3. Find y-intercept
4. Identify approximate turning points
5. Connect with smooth curves
6. Check with Desmos!

WATCH IT WORK

Example 1: Analyze and sketch $f(x) = x^3 - 4x$
Step 1: Factor to find zeros:

$$f(x) = x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$$

Zeros: $x = 0, 2, -2$
Step 2: End behavior:

- Degree 3 (odd), leading coefficient 1 (positive)
- As $x \to -\infty$, $f(x) \to -\infty$
- As $x \to +\infty$, $f(x) \to +\infty$

Step 3: y-intercept: $f(0) = 0$
Step 4: Turning points: At most 2 (since degree is 3)
Step 5: Test points between zeros:

- $f(-3) = (-3)(-3-2)(-3+2) = (-3)(-5)(-1) = -15$ (below x-axis)
- $f(1) = (1)(1-2)(1+2) = (1)(-1)(3) = -3$ (below x-axis)
- $f(3) = (3)(3-2)(3+2) = (3)(1)(5) = 15$ (above x-axis)

Desmos Verification:

1. Graph `y = x^3 - 4x`
2. Verify zeros at $x = -2, 0, 2$
3. Notice the S-shaped curve
4. Count the turning points — should be exactly 2

Example 2: Analyze $g(x) = -x^4 + 4x^2$
Step 1: Factor:

$$g(x) = -x^4 + 4x^2 = x^2(-x^2 + 4) = x^2(4 - x^2) = x^2(2 - x)(2 + x)$$

$$g(x) = x^2(4 - x^2) = x^2(2^2 - x^2) = x^2(2 - x)(2 + x)$$

Actually: $4 - x^2 = (2 - x)(2 + x) = -(x - 2)(x + 2)$ So: $g(x) = -x^2(x - 2)(x + 2)$
Zeros: $x = 0$ (multiplicity 2), $x = 2$, $x = -2$
Step 2: End behavior:

- Degree 4 (even), leading coefficient -1 (negative)
- As $x \to \pm\infty$, $g(x) \to -\infty$

Step 3: y-intercept: $g(0) = 0$
Step 4: At $x = 0$, the graph touches but doesn't cross (even multiplicity)
Desmos Exploration:

1. Graph `y = -x^4 + 4x^2`
2. Notice how it touches the x-axis at the origin but doesn't cross

YOUR TURN

1. **End Behavior Practice:** For each polynomial, predict the end behavior:
   a) $f(x) = 2x^3 - 5x + 1$
   b) $g(x) = -3x^4 + x^2 - 7$
   c) $h(x) = x^5 - 2x^3 + x$
   d) $j(x) = -x^6 + 4x^4$
2. **Graph Analysis with Desmos:**
   a) Graph `y = x^4 - 5x^2 + 4`
   b) How many x-intercepts do you see?
   c) How many turning points?
   d) What's the y-intercept?
   e) Describe the end behavior
3. **Sketch Challenge:** Before using Desmos, sketch a rough graph of $f(x) = (x+1)^2(x-3)$:
   a) Find all zeros and their multiplicities
   b) Determine end behavior
   c) Estimate turning points
   d) Draw your sketch
   e) Check with Desmos — how did you do?
4. **Creative Challenge:** Design a polynomial that:
   - Has exactly 3 x-intercepts
   - Goes to $+\infty$ as $x \to +\infty$
   - Goes to $-\infty$ as $x \to -\infty$
   - Has at most 2 turning points

CHECK YOUR VOICE

What did your brain say when analyzing polynomial behavior?
If it said *"This seems like a lot of rules to remember"* — remember, it's more about understanding the patterns than memorizing. Even degree = same behavior on both ends, odd degree = opposite behavior. Positive leading coefficient = upward, negative = downward.
If it said *"I can't picture this without graphing"* — that's perfectly fine! Visual learners often need to see the graph to understand the behavior. Use Desmos as your visual thinking partner.
If it said *"Some of these graphs look really complex"* — they do! But notice how each complex shape follows predictable rules. Once you understand the patterns, even 6th-degree polynomials become readable.
If it said *"This connects to everything we learned before"* — exactly! Polynomials are just linear and quadratic functions working as a team. All your previous knowledge still applies.

4.4 Real-World Polynomials (Where Teams of Linear Functions Show Up)

READ THIS FIRST

You're designing a rectangular garden against your house, so you only need to fence three sides.
You have 60 feet of fencing.
Let's say the width of the garden is x feet. Then the length must be $(60 - 2x)$ feet, because you need $x + (60 - 2x) + x = 60$ feet of fencing total.
The area of your garden is:

$$\text{Area} = \text{width} \times \text{length} = x(60 - 2x) = 60x - 2x^2$$

This is a quadratic polynomial! It represents how the area changes as you vary the width.
But gardens are just the beginning. Polynomials show up whenever you have:
- Multiple quantities multiplying together - Optimization problems (maximizing profit, minimizing cost) - Motion problems with acceleration - Population dynamics - Economic models - Engineering stress analysis
Every time real-world linear relationships interact with each other, polynomials emerge naturally.

LET'S TALK ABOUT IT

Think about situations where you're trying to optimize something:

- Planning a road trip — balancing time, distance, cost, and enjoyment
- Studying for multiple exams — allocating time for maximum overall performance
- Cooking for a party — scaling recipes while managing cost and preparation time
- Saving money — balancing current enjoyment with future financial security

How do multiple factors interact to affect the outcome you're trying to optimize?

NOW WE NAME IT

Real-World Polynomial Applications

1. Optimization Problems: When you're trying to maximize or minimize something subject to constraints, polynomials often emerge from the relationships between variables.

2. Business and Economics:

- **Revenue:** $R(x) = \text{(price)(quantity sold)}$
- **Profit:** $P(x) = R(x) - C(x)$ where $C(x)$ is cost
- **Market models:** Supply and demand interactions

3. Motion and Physics:

- **Position:** $s(t) = at^3 + bt^2 + ct + d$ (with acceleration)
- **Projectile motion:** Height as quadratic function of time
- **Oscillations:** Complex motion patterns

4. Population Dynamics:

- Growth models with environmental limits
- Predator-prey relationships
- Disease spread models

5. Engineering and Design:

- Structural stress analysis
- Fluid flow calculations
- Signal processing
- Circuit design

Problem-Solving Strategy:

1. Identify what you're trying to optimize
2. Define variables and constraints
3. Set up the polynomial relationship
4. Find critical points (zeros of the derivative)
5. Test to find maximum or minimum
6. Verify with real-world reasoning

WATCH IT WORK

Example 1: The Garden Optimization Problem
From our story: You have 60 feet of fencing for a garden against your house.
Step 1: Set up the problem

- Width = x feet
- Length = $(60 - 2x)$ feet
- Area = $A(x) = x(60 - 2x) = 60x - 2x^2$

Step 2: Find the domain What values of x make sense?

- $x > 0$ (positive width)
- $60 - 2x > 0 \Rightarrow x < 30$ (positive length)
- Domain: $0 < x < 30$

Step 3: Find maximum area using Desmos **Desmos Analysis:**

1. Graph `y = x(60 - 2x)` for `0 < x < 30`
2. Find the vertex of this parabola
3. The maximum occurs at $x = 15$ feet
4. Maximum area = $15(60 - 2 \cdot 15) = 15 \cdot 30 = 450$ square feet

Step 4: Interpret the result Optimal garden: 15 feet wide by 30 feet long, giving 450 square feet of area.

Example 2: Business Revenue Model
A company finds that when they price their product at p dollars, they sell $(100 - 2p)$ units.
Step 1: Set up revenue function

$$R(p) = (\text{price})(\text{quantity}) = p(100 - 2p) = 100p - 2p^2$$

Step 2: Find domain

- $p > 0$ (positive price)
- $100 - 2p > 0 \Rightarrow p < 50$ (positive quantity)
- Domain: $0 < p < 50$

Step 3: Find maximum revenue **Desmos Exploration:**

1. Graph `y = p(100 - 2p)` for `0 < p < 50`
2. Find the vertex: maximum at $p = 25$ dollars
3. Maximum revenue = $25(100 - 2 \cdot 25) = 25 \cdot 50 = 1250$ dollars

Step 4: Business insight Price the product at \$25 to maximize revenue of \$1,250.

Example 3: Projectile Motion
A ball is thrown upward from a 6-foot platform with initial velocity 32 ft/sec.
Height function: $h(t) = -16t^2 + 32t + 6$

YOUR TURN

1. **Fence Variation:** You have 100 feet of fencing to make a rectangular pen, fencing all four sides.

    a) If width $= x$, what is the length?

    b) Write the area function $A(x)$

    c) Use Desmos to find the dimensions that maximize area

    d) What's the maximum area?

2. **Business Problem:** A theater can sell 200 tickets at \$10 each. For every \$1 increase in price, they lose 10 customers.

    a) If price = \$(10 + x), how many tickets sell?

    b) Write the revenue function $R(x)$

    c) Use Desmos to find the optimal price

    d) What's the maximum revenue?

3. **Projectile Challenge:** A rocket is fired from ground level with initial velocity 80 ft/sec.

    a) Write the height function $h(t) = -16t^2 + 80t$

    b) Graph in Desmos

    c) When does it reach maximum height?

    d) What's the maximum height?

    e) When does it return to ground?

4. **Creative Application:** Design your own real-world polynomial problem involving:

    - A situation you care about
    - Two or more interacting linear relationships
    - An optimization goal
    - Analysis using Desmos

CHECK YOUR VOICE

What did your brain say when working with real-world polynomials?
If it said *"I didn't realize polynomials were in so many places"* — they're everywhere! Anytime you have quantities multiplying together or optimization problems, polynomials often emerge.
If it said *"Setting up the problem is harder than solving it"* — you're right! The mathematical modeling (translating real situations into math) is often the hardest part. But it's also the most valuable skill.
If it said *"Desmos makes this so much easier"* — absolutely! Being able to visualize the function helps you understand both the mathematics and the real-world situation better.
If it said *"I can see connections to business/physics/design now"* — this is mathematical maturity! You're recognizing that math isn't just abstract symbols — it's a language for describing how the world works.

YOUR TURN - EXTENDED PRACTICE

Chapter 4: Mastering Polynomial Functions

Problems That Bring Everything Together

1. Polynomial Family Tree
For each polynomial, identify the family characteristics:

a) $f(x) = (x-1)(x+2)(x-3)$

- Degree: ___________
- Zeros: ______________________
- End behavior: ______________________
- Maximum turning points: ___________

b) $g(x) = -2x^4 + 8x^2$

- Factor completely: ______________________
- Zeros with multiplicities: ______________________
- End behavior: ______________________
- y-intercept: ___________

c) $h(x) = x^5 - 16x^3$

- Factor completely: ______________________
- Degree: ___________
- Number of x-intercepts: ___________
- End behavior: ______________________

2. Factor Detective Challenge
Factor completely and verify with Desmos:

a) $f(x) = x^3 - 9x$

b) $g(x) = x^4 - 10x^2 + 9$

c) $h(x) = 2x^3 + 6x^2 - 8x - 24$

d) $j(x) = x^4 - 16$

3. Graph Sketching Championship
For each polynomial, sketch a rough graph BEFORE using Desmos, then check:

a) $f(x) = x^3 - 4x^2 + 3x$

- Predicted zeros: ______________________
- Predicted end behavior: ______________________
- Your sketch: [Draw rough sketch]
- Desmos accuracy check: ______________________

CHAPTER 4 COMPLETE!

You've now mastered the extended polynomial family! You understand that polynomials are just teams of linear functions working together, and you can:

Factor Detective Skills: Break polynomials back into their linear team members

Graph Reading: Predict polynomial behavior from degree and leading coefficient

Real-World Modeling: Recognize polynomial relationships in business, physics, and optimization

Visual Confidence: Use Desmos to explore, verify, and discover polynomial patterns

Most importantly: You see polynomials not as intimidating high-degree expressions, but as collaborative mathematical personalities with predictable behaviors and real-world applications.

Ready for Chapter 5? We're about to explore "The Transformation Artists" — how functions can shift, stretch, reflect, and compose with each other to create even more sophisticated mathematical relationships!

Your mathematical toolkit keeps growing stronger!

ANSWER KEY - CHAPTER 4

Problem 1: Polynomial Family Tree

a) $f(x) = (x-1)(x+2)(x-3)$

- Degree: 3 (cubic)
- Zeros: $x = 1, -2, 3$
- End behavior: $x \to -\infty, f(x) \to -\infty$; $x \to +\infty, f(x) \to +\infty$
- Maximum turning points: 2

b) $g(x) = -2x^4 + 8x^2 = -2x^2(x^2-4) = -2x^2(x-2)(x+2)$

- Factor completely: $-2x^2(x-2)(x+2)$
- Zeros with multiplicities: $x = 0$ (mult. 2), $x = 2, -2$ (mult. 1 each)
- End behavior: Both directions go to $-\infty$ (even degree, negative leading coeff)
- y-intercept: $(0, 0)$

c) $h(x) = x^5 - 16x^3 = x^3(x^2-16) = x^3(x-4)(x+4)$

- Factor completely: $x^3(x-4)(x+4)$
- Degree: 5
- Number of x-intercepts: 3 (at $x = 0, 4, -4$)
- End behavior: $x \to -\infty, f(x) \to -\infty$; $x \to +\infty, f(x) \to +\infty$

Problem 2: Factor Detective Challenge

a) $f(x) = x^3 - 9x = x(x^2-9) = x(x-3)(x+3)$

b) $g(x) = x^4 - 10x^2 + 9$. Let $u = x^2$: $u^2 - 10u + 9 = (u-1)(u-9) = (x^2-1)(x^2-9) = (x-1)(x+1)(x-3)(x+3)$

c) $h(x) = 2x^3 + 6x^2 - 8x - 24 = 2(x^3 + 3x^2 - 4x - 12)$ Group: $= 2[x^2(x+3) - 4(x+3)] = 2(x+3)(x^2-4) = 2(x+3)(x-2)(x+2)$

d) $j(x) = x^4 - 16 = (x^2)^2 - 4^2 = (x^2-4)(x^2+4) = (x-2)(x+2)(x^2+4)$

Detective insight: Notice how factoring often involves recognizing patterns (difference of squares, grouping) and sometimes substitution ($u = x^2$). Each factoring method is a tool in your mathematical detective kit!

Problem 3: Graph Sketching Championship

a) $f(x) = x^3 - 4x^2 + 3x = x(x^2 - 4x + 3) = x(x-1)(x-3)$

- Predicted zeros: $x = 0, 1, 3$
- Predicted end behavior: $x \to -\infty, f(x) \to -\infty$; $x \to +\infty, f(x) \to +\infty$
- Desmos shows: S-curve passing through these three points with 2 turning points

b) $g(x) = -(x+1)^2(x-2)^2$

- Predicted zeros and multiplicities: $x = -1$ (mult. 2), $x = 2$ (mult. 2)
- Predicted end behavior: Both directions go to $-\infty$ (even degree, negative leading coeff)
- Special features: Graph touches x-axis at both zeros but doesn't cross (even multiplicities)
- Desmos shows: W-shaped curve opening downward

Sketching success: If your predictions matched Desmos within reason, congratulations! You're thinking like a polynomial artist. If not, that's valuable learning — notice what you missed and why.

Problem 4: Real-World Polynomial Modeling - The Package Problem

a) Cutting squares of side length x from each corner:

- Length of box: $20 - 2x$ inches
- Width of box: $12 - 2x$ inches
- Height of box: x inches

b) Volume function: $V(x) = x(20 - 2x)(12 - 2x) = x(240 - 40x - 24x + 4x^2) = 4x^3 - 64x^2 + 240x$

c) Domain: $0 < x < 6$ (width constraint: $12 - 2x > 0$)

d) Desmos results:

- Value of x that maximizes volume: $x \approx 2.13$ inches
- Maximum volume: ≈ 228 cubic inches
- Optimal box dimensions: 15.74" $\times$ 7.74" $\times$ 2.13"

Real-world insight: This cubic polynomial shows that optimization often involves balancing competing factors — cutting larger squares gives more height but reduces the base area.

Problem 5: Advanced Business Application
For $P(x) = -2x^3 + 27x^2 - 84x + 45$:

a) Desmos graph shows a cubic with negative leading coefficient

b) Break-even points (where $P(x) = 0$): $x \approx 0.6, 3.0, 9.9$ thousand units

c) Positive profit regions: Between $x \approx 0.6$ and $x \approx 3.0$, and between $x \approx 3.0$ and $x \approx 9.9$

d) Maximum profit occurs at approximately $x \approx 6$ thousand units

e) Maximum profit: ≈ 135 thousand dollars

f) Negative profit regions mean the company loses money at very low production (high fixed costs) or very high production (market saturation, increased costs)

Business wisdom: This shows why real businesses must find their "sweet spot" — too little or too much production both lead to losses.

Problem 6: Polynomial Engineering

a) Degree 4, zeros at $x = -2, 0, 1, 3$, y-intercept at $(0, 0)$: $f(x) = ax(x + 2)(x - 1)(x - 3)$ where a can be any non-zero constant Example: $f(x) = x(x + 2)(x - 1)(x - 3)$

b) Degree 3, one real zero, goes from $-\infty$ to $+\infty$: Need positive leading coefficient and complex zeros. Example: $f(x) = (x - 1)(x^2 + 1) = x^3 - x^2 + x - 1$

c) Even degree, two x-intercepts, opens downward: Example: $f(x) = -(x - 1)^2(x + 2)^2$ or $f(x) = -(x^2 - 4)$

d) Odd degree, three x-intercepts, one with multiplicity 2: Example: $f(x) = (x-1)^2(x+2)$ (degree 3, zeros at $x = 1$ [mult. 2] and $x = -2$)

Engineering insight: You can design polynomials to meet specific criteria by understanding how zeros, multiplicity, and leading coefficient affect the graph!

CHECK YOUR VOICE - AFTER CHAPTER 4

What did your brain say as you completed Chapter 4?
If it said *"Polynomials aren't as scary as I thought"* — that's the breakthrough! When you see polynomials as teams of linear functions, they become manageable and even interesting.
If it said *"I can predict polynomial behavior now"* — you've developed mathematical intuition! You're seeing the patterns that govern how these functions behave.
If it said *"Factoring feels like detective work"* — exactly! You're uncovering hidden structure and seeing mathematical relationships that were always there.
If it said *"Real-world applications make this meaningful"* — mathematical maturity! You're seeing how abstract concepts connect to problems that actually matter.
If it said *"I want to explore more complex polynomials"* — that's mathematical curiosity awakening! You're ready to tackle even more sophisticated mathematical ideas.
You've come so far from thinking "I'm not a math person" to confidently analyzing cubic and quartic polynomials!
Ready for Chapter 5: The Transformation Artists? We're about to discover how functions can shift, stretch, flip, and compose with each other to create mathematical magic!

Chapter 5

THE TRANSFORMATION ARTISTS

Function Composition & Inverse Functions

SKILLS CHECK: What You Need From Previous Chapters

- **Function Notation:** You're comfortable with $f(x)$, $g(x)$, and evaluating functions
- **Function Families:** You recognize linear, quadratic, absolute value, and polynomial personalities
- **Coordinate Plane:** You can read and interpret function graphs confidently
- **Function Arithmetic:** You can add, subtract, multiply, and divide functions
- **Desmos Mastery:** You're fluent in visual mathematical exploration

If any of these feel uncertain, a quick review of previous chapters will help!

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll think of functions as **transformation artists** who can shift, stretch, reflect, combine, and even "undo" each other to create infinitely sophisticated mathematical relationships.

The Story: You'll discover that transformations you see every day (moving, resizing, flipping, layering) are the same mathematical operations that create advanced function behaviors.

The Skills You'll Have:

- Transform function graphs by shifting, stretching, and reflecting
- Compose functions (put one function inside another)
- Find and verify inverse functions (mathematical "undo" operations)
- Recognize transformation patterns in real-world applications
- Use transformations to understand complex function relationships

The Confidence Moment: When you see $f(g(x))$ or $f^{-1}(x)$, your brain will think "oh, that's function composition" or "that's the inverse function" instead of "scary advanced notation I don't understand."

The Bridge: Chapter 6 will show you exponential and logarithmic functions, which are the ultimate examples of functions and their inverses working together to model growth and decay.

5.1 Function Transformations (Mathematical Shapeshifting)

READ THIS FIRST

You're editing a photo on your phone.
You start with a basic image of your friend. But that's just the beginning.
You can **move** the image around the screen — slide it left, right, up, down.
You can **resize** it — make it taller, wider, bigger, smaller.
You can **flip** it — create a mirror image horizontally or vertically.
Each of these operations takes your original image and transforms it into something new, but you can still recognize the essential "friend-ness" underneath all the changes.
The same thing happens with function graphs.
You start with a basic function like $f(x) = x^2$ — that's your original "photo."
Then you can transform it: - $f(x) + 3$ moves it up - $f(x - 2)$ moves it right - $2f(x)$ stretches it vertically - $-f(x)$ flips it upside down
Every transformation preserves the essential "parabola-ness" while creating a new mathematical personality.
And just like with photo editing, once you learn the basic moves, you can combine them to create incredibly sophisticated results.

LET'S TALK ABOUT IT

Think about transformations you do in everyday life:

- Rearranging furniture — moving, rotating, resizing to fit different spaces
- Adjusting recipes — scaling ingredients up or down for different serving sizes
- Editing photos or videos — cropping, resizing, filtering, adjusting brightness
- Playing music — changing tempo, pitch, volume, or adding effects

What patterns do you notice when you transform things? What stays the same, and what changes?

NOW WE NAME IT

Function Transformations are operations that change the position, size, or orientation of a function's graph while preserving its basic shape and personality.

Basic Transformation Types:

1. Vertical Shifts:

- $f(x) + k$ shifts the graph up by k units (if $k > 0$) or down by $|k|$ units (if $k < 0$)
- Every point (x, y) becomes $(x, y + k)$

2. Horizontal Shifts:

- $f(x - h)$ shifts the graph right by h units (if $h > 0$) or left by $|h|$ units (if $h < 0$)
- Every point (x, y) becomes $(x + h, y)$
- **Caution:** The sign is opposite to what you might expect!

3. Vertical Stretching/Compressing:

- $a \cdot f(x)$ where $|a| > 1$ stretches vertically, $0 < |a| < 1$ compresses vertically
- Every point (x, y) becomes $(x, a \cdot y)$

4. Horizontal Stretching/Compressing:

- $f(b \cdot x)$ where $|b| > 1$ compresses horizontally, $0 < |b| < 1$ stretches horizontally
- Every point (x, y) becomes $(\frac{x}{b}, y)$
- **Caution:** Again, the effect is opposite to what you might expect!

5. Reflections:

- $-f(x)$ reflects across the x-axis (flips upside down)
- $f(-x)$ reflects across the y-axis (flips left to right)

Multiple Transformations: The general form is $f(x) = a \cdot g(b(x - h)) + k$ where:

- a controls vertical stretch/compression and vertical reflection
- b controls horizontal stretch/compression and horizontal reflection
- h controls horizontal shift
- k controls vertical shift

WATCH IT WORK

Example 1: Transform $f(x) = x^2$ step by step
Starting point: $f(x) = x^2$ (basic parabola with vertex at origin)
Step 1: Apply $f(x) + 3$

- Result: $g(x) = x^2 + 3$
- Effect: Shifts the entire parabola up 3 units
- New vertex: $(0, 3)$

Step 2: Apply $f(x - 2)$ to our original function

- Result: $h(x) = (x - 2)^2$
- Effect: Shifts the parabola right 2 units
- New vertex: $(2, 0)$

Step 3: Combine both transformations

- Result: $j(x) = (x - 2)^2 + 3$
- Effect: Shifts right 2 units AND up 3 units
- New vertex: $(2, 3)$

Desmos Exploration:

1. Graph `y = x^2` (original)
2. Add `y = x^2 + 3` (vertical shift)
3. Add `y = (x - 2)^2` (horizontal shift)
4. Add `y = (x - 2)^2 + 3` (combined)
5. Watch how each transformation affects the graph!

Example 2: Explore stretching and reflecting
Starting with: $f(x) = |x|$ (basic V-shape)
Vertical stretch: $g(x) = 3|x|$

- Every y-value gets multiplied by 3
- The V becomes "skinnier" (steeper)
- Vertex stays at $(0, 0)$

Vertical reflection: $h(x) = -|x|$

- Every y-value becomes negative
- The V flips upside down (becomes an upside-down V)
- Vertex stays at $(0, 0)$

Combined: $j(x) = -3|x|$

- Upside-down V that's stretched (steeper)

YOUR TURN

1. **Transformation Identification:** Describe how each function transforms $f(x) = x^2$:
   a) $g(x) = x^2 - 5$
   b) $h(x) = (x + 3)^2$
   c) $j(x) = 4x^2$
   d) $k(x) = -(x - 2)^2 + 1$
2. **Desmos Transformation Lab:** Start with $f(x) = |x|$ and create these transformations:
   a) Shift right 4 units
   b) Shift down 2 units
   c) Stretch vertically by factor of 3
   d) Combine all three transformations
3. **Reverse Engineering:** Given the transformed function, identify the original and the transformations:
   a) $f(x) = 3(x - 1)^2 + 2$ (starts from $y = x^2$)
   b) $g(x) = -|x + 4| - 1$ (starts from $y = |x|$)
   c) $h(x) = 2x + 6$ (starts from $y = x$)
4. **Creative Challenge:** Use transformations to move $f(x) = x^2$ so that:
   - Its vertex is at $(3, -2)$
   - It opens downward
   - It's stretched vertically by a factor of 1.5

CHECK YOUR VOICE

What did your brain say when exploring function transformations?
If it said *"This is like moving puzzle pieces around"* — that's exactly right! Transformations are systematic ways to reposition and reshape mathematical objects.
If it said *"I keep getting the horizontal shifts backwards"* — you're not alone! $f(x-2)$ moving right (not left) is counterintuitive. Think of it as "what value of x makes $(x - 2) = 0$?" The answer is $x = 2$, so the vertex moves to $x = 2$.
If it said *"Desmos makes this so much clearer"* — visual learning is powerful! Seeing the transformations happen in real-time helps your brain understand the pattern.
If it said *"I can predict what transformations will do now"* — that's mathematical intuition developing! You're seeing the systematic nature of how functions change.

5.2 Function Composition (Mathematical Nesting)

READ THIS FIRST

You're getting ready for work in the morning.
You have a routine: 1. First, you take a shower (function g) 2. Then, you get dressed (function f)
But here's the thing — you can't do step 2 without first doing step 1. Getting dressed depends on having taken a shower. In mathematical terms, you're doing $f(g(\text{you}))$.
The "output" of g (clean you) becomes the "input" for f (dressed you).
This is function composition — when one function takes the output of another function as its input.
Or think about your phone's camera app: 1. First, it takes the picture (function g: reality → digital image) 2. Then, it applies a filter (function f: digital image → filtered image)
The final result is $f(g(\text{reality}))$ — a filtered version of the original photo.
Composition happens everywhere: cooking (prep → cook → serve), GPS navigation (current location → calculate route → give directions), even ordering coffee (choose drink → customize → pay → receive).
In each case, you're putting one process inside another process to create a more complex but useful operation.

LET'S TALK ABOUT IT

Think about everyday activities that involve steps that depend on each other:

- Making breakfast — getting ingredients (step 1) before cooking (step 2)
- Online shopping — selecting items (step 1) before checkout (step 2)
- Learning a skill — understanding basics (step 1) before advanced techniques (step 2)
- Travel planning — choosing destination (step 1) before booking flights (step 2)

What happens if you try to do step 2 before step 1? How do the steps connect to each other?

NOW WE NAME IT

Function Composition is the process of applying one function to the output of another function.
Notation and Definition:

$$(f \circ g)(x) = f(g(x))$$

Read as: "f composed with g" or "f of g of x"
How it Works:

1. Start with input x
2. Apply function g to get $g(x)$
3. Apply function f to the result: $f(g(x))$

Important Notes:

- Order matters! $f(g(x)) \neq g(f(x))$ in general
- Think "inside out" — work from the innermost function outward
- The output of the inner function must be in the domain of the outer function

Domain of Composite Functions: The domain of $f(g(x))$ consists of all x values where:

1. x is in the domain of g, AND
2. $g(x)$ is in the domain of f

Real-World Examples:

- Temperature conversion: Celsius → Kelvin → Fahrenheit
- Business: Raw materials → Manufacturing → Retail price
- Photography: Scene → Camera → Filter → Display
- Math: Input → First calculation → Second calculation → Output

Visual Understanding: Think of composition as a "function machine factory" where the output of one machine immediately feeds into the next machine.

WATCH IT WORK

Example 1: Basic function composition
Given: $f(x) = 2x + 1$ and $g(x) = x^2$
Find $(f \circ g)(x) = f(g(x))$*:*
Step 1: Start with $g(x) = x^2$
Step 2: Apply f to this result: $f(g(x)) = f(x^2)$
Step 3: Substitute x^2 into the f function:

$$f(x^2) = 2(x^2) + 1 = 2x^2 + 1$$

Result: $(f \circ g)(x) = 2x^2 + 1$
Find $(g \circ f)(x) = g(f(x))$*:*
Step 1: Start with $f(x) = 2x + 1$
Step 2: Apply g to this result: $g(f(x)) = g(2x + 1)$
Step 3: Substitute $(2x + 1)$ into the g function:

$$g(2x + 1) = (2x + 1)^2 = 4x^2 + 4x + 1$$

Result: $(g \circ f)(x) = 4x^2 + 4x + 1$
Key observation: $f(g(x)) \neq g(f(x))$ — order matters!
Desmos Verification:

1. Define `f(x) = 2x + 1` and `g(x) = x^2`
2. Graph `y = f(g(x))` which should give `y = 2x^2 + 1`
3. Graph `y = g(f(x))` which should give `y = (2x + 1)^2`
4. Compare the graphs — they're different!

Example 2: Evaluating composition at specific points
Using the same functions: $f(x) = 2x + 1$ and $g(x) = x^2$
Find $(f \circ g)(3)$*:*
Method 1 — Direct substitution:

$$\begin{aligned}
(f \circ g)(3) &= f(g(3)) && (5.1)\\
&= f(3^2) && (5.2)\\
&= f(9) && (5.3)\\
&= 2(9) + 1 && (5.4)\\
&= 19 && (5.5)
\end{aligned}$$

Method 2 — Use the composition formula:

$$\begin{aligned}
(f \circ g)(x) &= 2x^2 + 1 && (5.6)\\
(f \circ g)(3) &= 2(3)^2 + 1 = 18 + 1 = 19 && (5.7)
\end{aligned}$$

Both methods give the same result!

Example 3: Real-world composition

A company's profit depends on sales, and sales depend on advertising.
Let a = advertising dollars spent (in thousands) Let $s(a) = 5a + 10$ = number of sales generated Let $p(s) = 0.3s - 2$ = profit (in thousands) from s sales
Question: What's the profit as a function of advertising?

YOUR TURN

1. **Basic Composition:** Given $f(x) = x + 3$ and $g(x) = 2x$:
   a) Find $(f \circ g)(x)$
   b) Find $(g \circ f)(x)$
   c) Are they the same? Why or why not?
   d) Calculate $(f \circ g)(5)$ two different ways
2. **Desmos Composition Exploration:**
   a) Let `f(x) = sqrt(x)` and `g(x) = x - 4`
   b) Graph `y = f(g(x))`
   c) What's the domain of this composition? Why?
   d) Try `y = g(f(x))` — how is it different?
3. **Real-World Composition:** A pizza restaurant's model:
   - Temperature affects customers: $c(t) = -2t+120$ (customers when the temperature is t degrees)
   - Customers affect revenue: $r(c) = 12c$ (revenue from c customers)

   a) Find the revenue as a function of temperature: $(r \circ c)(t)$
   b) What's the revenue when it's 70°F?
   c) At what temperature is revenue maximized? (Hint: think about realistic temperatures!)
4. **Decomposition Challenge:** Express $h(x) = \sqrt{3x + 1}$ as a composition of two simpler functions f and g.

CHECK YOUR VOICE

What did your brain say when working with function composition?
If it said *"This is like following a recipe with steps"* — perfect analogy! Composition is about doing operations in the right order.
If it said *"I keep getting confused about which function goes where"* — remember, $f(g(x))$ means g happens first (inner function), then f happens to that result. Work from the inside out.
If it said *"The real-world examples make more sense"* — that's because composition naturally describes how complex processes work in steps. Math is modeling the way the world actually operates.
If it said *"Domain issues are tricky"* — you're thinking deeply! The output of the first function has to be a valid input for the second function. That's good mathematical reasoning.

5.3 Inverse Functions (Mathematical Undo Operations)

READ THIS FIRST

You're setting up a password for your bank account.
The bank's security system takes your password "fluffy123" and runs it through an encryption function. Let's call this function f.
f(fluffy123) = x7b9k2m4z8 (encrypted version)
Now, when you log in tomorrow, you type "fluffy123" again. The system needs to check if it matches what they have stored.
But they only have the encrypted version "x7b9k2m4z8". So they need a different function — let's call it f^{-1} — that can undo the encryption:
f^{-1}(x7b9k2m4z8) = fluffy123 (decrypted version)
Function f^{-1} is the **inverse** of function f. It undoes exactly what f did.
This "undo" relationship is everywhere: - Putting on shoes ↔ Taking off shoes - Locking a door ↔ Unlocking a door - Converting Celsius to Fahrenheit ↔ Converting Fahrenheit to Celsius - Saving money ↔ Spending money - Encryption ↔ Decryption
In mathematics, inverse functions are "undo" operations. If f takes you from a to b, then f^{-1} takes you from b back to a.
The beautiful thing? When you do a function and its inverse in sequence, you get back exactly where you started: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
It's like putting on your shoes and then immediately taking them off — you're back to barefoot.

LET'S TALK ABOUT IT

Think about everyday "undo" operations in your life:

- Getting dressed in the morning, then undressing at night
- Driving to work, then driving home (sort of — same route, opposite direction)
- Making coffee and then drinking it (okay, that one doesn't undo...)
- Sending a text and then... well, you can't really unsend it

What makes some operations "undoable" and others not? When can you perfectly reverse a process?

NOW WE NAME IT

An **Inverse Function** f^{-1} is a function that "undoes" what function f does.
Formal Definition: Functions f and f^{-1} are inverses if:

$$f(f^{-1}(x)) = x \quad \text{for all } x \text{ in domain of } f^{-1} \tag{5.12}$$

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in domain of } f \tag{5.13}$$

Key Properties:

- If (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}
- The graphs of f and f^{-1} are reflections of each other across the line $y = x$
- Domain of f = Range of f^{-1}
- Range of f = Domain of f^{-1}

When Does an Inverse Function Exist? A function has an inverse if and only if it's **one-to-one** (also called **injective**):

- Each output comes from exactly one input
- Passes the **Horizontal Line Test**: any horizontal line intersects the graph at most once
- No two different inputs give the same output

Finding Inverse Functions:

1. Replace $f(x)$ with y: $y = f(x)$
2. Swap x and y: $x = f(y)$
3. Solve for y: $y = f^{-1}(x)$
4. Verify: Check that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Notation Warning: $f^{-1}(x) \neq \frac{1}{f(x)}$
The -1 is not an exponent! It's a special notation meaning "inverse function."

WATCH IT WORK

Example 1: Find the inverse of a linear function
Given: $f(x) = 2x + 3$
Step 1: Replace $f(x)$ with y:

$$y = 2x + 3$$

Step 2: Swap x and y:

$$x = 2y + 3$$

Step 3: Solve for y:

$$x = 2y + 3 \tag{5.14}$$
$$x - 3 = 2y \tag{5.15}$$
$$\frac{x-3}{2} = y \tag{5.16}$$

Step 4: Write the inverse:

$$f^{-1}(x) = \frac{x-3}{2}$$

Verification:

$$f(f^{-1}(x)) = f\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3 = (x-3) + 3 = x \quad \checkmark \tag{5.17}$$
$$f^{-1}(f(x)) = f^{-1}(2x+3) = \frac{(2x+3)-3}{2} = \frac{2x}{2} = x \quad \checkmark \tag{5.18}$$

Desmos Visualization:

1. Graph `y = 2x + 3` (original function)
2. Graph `y = (x - 3)/2` (inverse function)
3. Add the line `y = x`
4. Notice how f and f^{-1} are mirror images across $y = x$!

Example 2: Check if a function has an inverse
Consider: $g(x) = x^2$
Horizontal Line Test: The line $y = 4$ intersects the parabola $y = x^2$ at two points: $(-2, 4)$ and $(2, 4)$.
This means $g(-2) = g(2) = 4$ — two different inputs give the same output.
Conclusion: $g(x) = x^2$ is NOT one-to-one, so it doesn't have an inverse function over all real numbers.
But wait... If we restrict the domain to $x \geq 0$, then $g(x) = x^2$ for $x \geq 0$ IS one-to-one!
Restricted function: $g(x) = x^2$ for $x \geq 0$ *Its inverse:* $g^{-1}(x) = \sqrt{x}$ for $x \geq 0$
Desmos Investigation:

1. Graph `y = x^2`
2. Try horizontal line test — fails!
3. Graph `y = x^2` for `x >= 0`
4. Graph `y = sqrt(x)`
5. Add `y = x` — see the reflection relationship!

YOUR TURN

1. **Finding Inverses:** Find the inverse of each function:
   a) $f(x) = 3x - 7$
   b) $g(x) = \frac{x+4}{2}$
   c) $h(x) = x^3$ (this one works for all real numbers!)
2. **Verification Practice:** For $f(x) = 4x + 1$ and $f^{-1}(x) = \frac{x-1}{4}$:
   a) Show that $f(f^{-1}(x)) = x$
   b) Show that $f^{-1}(f(x)) = x$
   c) Test with a specific value: calculate $f(5)$, then apply f^{-1} to that result
3. **One-to-One Investigation:** Use Desmos to determine which functions have inverses:
   a) $f(x) = x^3 - 2$
   b) $g(x) = |x|$
   c) $h(x) = 2^x$
   d) $j(x) = \sin(x)$ (hint: try restricting the domain)
4. **Real-World Application:** A store's pricing function is $P(c) = 2c + 15$, where c is the cost and P is the price.
   a) Find the inverse function $P^{-1}(P)$
   b) What does the inverse function represent?
   c) If an item is priced at \$35, what did it cost the store?

CHECK YOUR VOICE

What did your brain say when working with inverse functions?
If it said *"This is like figuring out how to undo something"* — exactly! Inverse functions are mathematical "undo" operations.
If it said *"The horizontal line test makes sense now"* — you're connecting visual and algebraic understanding! Functions need to be one-to-one to have inverses.
If it said *"Swapping variables feels weird"* — that's normal! The swap step is the mathematical way of asking "if this was the output, what was the input?"
If it said *"Some functions don't have inverses"* — great observation! Not everything can be undone. That's why we have the one-to-one requirement.
If it said *"The reflection across $y = x$ is beautiful"* — you're seeing the deep geometric meaning! Inverse functions are literally reflections of each other.

YOUR TURN - EXTENDED PRACTICE

Chapter 5: Mastering Function Transformations

Transformation Mastery

1. Transformation Recognition Championship
For each function, identify ALL transformations applied to the parent function:

a) $f(x) = -2(x+3)^2 - 1$ (parent: $y = x^2$)

- Transformations: ____
- New vertex: _______________
- Opens: _______________

b) $g(x) = 3|x-4| + 2$ (parent: $y = |x|$)

- Transformations: ____
- New vertex: _______________
- Shape changes: _______________

c) $h(x) = \frac{1}{2}(x+1)^3 - 4$ (parent: $y = x^3$)

- Transformations: ____
- Key point movement: $(0,0) \to$ _______________

2. Transformation Engineering
Create functions with these specifications:

a) Transform $y = x^2$ to have vertex at $(-2, 5)$ and open downward

b) Transform $y = |x|$ to be stretched vertically by factor 3, shifted right 1, down 2

c) Transform $y = \sqrt{x}$ to be reflected across x-axis and shifted left 4 units

d) Create a parabola that passes through points $(1, 0)$, $(3, 0)$, and has vertex at $(2, -1)$

3. Composition Challenges
Given $f(x) = x^2 + 1$, $g(x) = 2x - 3$, and $h(x) = \sqrt{x}$:

a) Find $(f \circ g)(x)$ and $(g \circ f)(x)$

b) Calculate $(f \circ g)(4)$ two different ways

c) Find $(h \circ f)(x)$ and determine its domain

d) Express $F(x) = \sqrt{2x^2 + 2 - 3}$ as a composition of the given functions

e) Create a real-world scenario where $(g \circ f)(x)$ would be meaningful

4. Inverse Function Mastery

a) Find the inverse of $f(x) = \frac{2x-1}{\text{[illegible]}}$ and verify your answer

ANSWER KEY - CHAPTER 5

Problem 1: Transformation Recognition Championship

a) $f(x) = -2(x+3)^2 - 1$ (parent: $y = x^2$)

- Transformations: Left 3, reflect over x-axis, vertical stretch by 2, down 1
- New vertex: $(-3, -1)$
- Opens: Downward

b) $g(x) = 3|x-4| + 2$ (parent: $y = |x|$)

- Transformations: Right 4, vertical stretch by 3, up 2
- New vertex: $(4, 2)$
- Shape changes: Steeper V-shape

c) $h(x) = \frac{1}{2}(x+1)^3 - 4$ (parent: $y = x^3$)

- Transformations: Left 1, vertical compression by 1/2, down 4
- Key point movement: $(0, 0) \to (-1, -4)$

Problem 2: Transformation Engineering

a) Vertex at $(-2, 5)$, opens downward: $f(x) = -(x+2)^2 + 5$

b) Vertical stretch by 3, right 1, down 2: $g(x) = 3|x-1| - 2$

c) Reflect over x-axis, left 4: $h(x) = -\sqrt{x+4}$

d) Passes through $(1, 0)$, $(3, 0)$, vertex $(2, -1)$: $f(x) = (x-1)(x-3) = x^2 - 4x + 3$ or vertex form: $f(x) = (x-2)^2 - 1$

Problem 3: Composition Challenges
Given $f(x) = x^2 + 1$, $g(x) = 2x - 3$, $h(x) = \sqrt{x}$:

a) $(f \circ g)(x) = f(g(x)) = (2x-3)^2 + 1 = 4x^2 - 12x + 10$
$(g \circ f)(x) = g(f(x)) = 2(x^2+1) - 3 = 2x^2 - 1$

b) $(f \circ g)(4) = 4(16) - 12(4) + 10 = 64 - 48 + 10 = 26$
Method 2: $g(4) = 5$, then $f(5) = 26$ ✓

c) $(h \circ f)(x) = \sqrt{x^2+1}$, Domain: all real numbers (since $x^2 + 1 \geq 1 > 0$)

d) $F(x) = \sqrt{2x^2 + 2 - 3} = \sqrt{2x^2 - 1} = h(g(f(x)) - 4)$
Actually: $F(x) = h((g \circ f)(x) - 2) = h(2x^2 - 1 - 2) = h(2x^2 - 3)$. This doesn't work neatly with the given functions.

e) Real-world: If $f(x)$ represents hours worked and gives points earned, and $g(x)$ converts points to dollars, then $(g \circ f)(x)$ gives salary based on hours worked.

Problem 4: Inverse Function Mastery

a) $f(x) = \frac{2x-1}{3}$. Inverse: $y = \frac{2x-1}{3} \Rightarrow 3y = 2x - 1 \Rightarrow x = \frac{3y+1}{2}$
So $f^{-1}(x) = \frac{3x+1}{2}$

b) Functions with inverses:

- $f(x) = x^4$: No (fails horizontal line test)
- $g(x) = x^3 - 2x$: No (has local max/min, fails horizontal line test)
- $h(x) = e^x$: Yes (always increasing)
- $j(x) = \frac{1}{x}$: Yes (strictly decreasing on each part of domain)

c) For $f(x) = (x-2)^2 + 3$, $x \geq 2$:
$y = (x-2)^2 + 3 \Rightarrow y - 3 = (x-2)^2 \Rightarrow x - 2 = \sqrt{y-3} \Rightarrow x = 2 + \sqrt{y-3}$
So $f^{-1}(x) = 2 + \sqrt{x-3}$, Domain: $x \geq 3$

d) If $g(3) = 7$, then $g^{-1}(7) = 3$

Problem 5: Food Truck Business Model

a) $(p \circ r \circ c)(t) = p(r(c(t))) = p(8(-0.5t^2 + 40t - 500)) = 8(-0.5t^2 + 40t - 500) - 200$
$= -4t^2 + 320t - 4000 - 200 = -4t^2 + 320t - 4200$

b) Graph shows parabola opening downward

c) Maximum at $t = -\frac{320}{2(-4)} = 40$°F

d) Maximum profit: $-4(40)^2 + 320(40) - 4200 = -6400 + 12800 - 4200 = \$2200$

e) Break even when profit = 0: $-4t^2 + 320t - 4200 = 0$
$t^2 - 80t + 1050 = 0$. Using quadratic formula: $t \approx 15$°F and 65°F

f) For \$300 profit: $-4t^2 + 320t - 4200 = 300$
$-4t^2 + 320t - 4500 = 0$, so $t \approx 18.4$°F or 61.6°F

CHECK YOUR VOICE - AFTER CHAPTER 5

What did your brain say as you mastered function transformations?
If it said *"Transformations are like photo editing for math"* — that's the perfect analogy! You've connected mathematical operations to familiar visual processes.
If it said *"Function composition makes sense when I think of it as steps"* — exactly! You're seeing how complex processes break down into simple, ordered operations.
If it said *"Inverse functions are mathematical undo buttons"* — brilliant insight! You understand that mathematical operations can be reversed systematically.
If it said *"I can predict what transformations will do before graphing"* — mathematical intuition! You're developing the ability to visualize mathematical relationships.
If it said *"Real-world problems use all these concepts together"* — mathematical maturity! You're seeing how abstract concepts combine to model complex situations.
You've transformed from someone who might have thought function notation was scary to someone who can manipulate, compose, and invert functions with confidence!
Ready for Chapter 6: The Growth Experts? We're about to explore exponential and logarithmic functions — the ultimate examples of inverse functions working together to model everything from bacterial growth to earthquake intensity!

CHAPTER 5 COMPLETE!

You've now mastered the transformation arts! You understand that functions are shapeshifters who can:

Transform with Purpose: Shift, stretch, reflect, and combine to create exactly the mathematical relationships you need

Compose Sophisticatedly: Nest functions inside each other to model complex multi-step processes

Undo with Precision: Find inverse functions that perfectly reverse mathematical operations

Visualize Confidently: Use Desmos to explore, predict, and verify transformation behaviors

Model Realistically: Recognize how transformations appear in business, science, and everyday applications

Most importantly: You see functions not as static equations, but as dynamic mathematical artists capable of infinite creativity and practical usefulness.

Ready for Chapter 6? We're about to meet "The Growth Experts" — exponential and logarithmic functions! These are the ultimate examples of inverse functions working together to model everything from population growth to radioactive decay to compound interest.

Your mathematical transformation powers are now complete!

Chapter 6

THE GROWTH EXPERTS

Exponential & Logarithmic Functions

SKILLS CHECK: What You Need From Previous Chapters

- **Function Transformations:** You can shift, stretch, and reflect function graphs confidently
- **Inverse Functions:** You understand the "undo" relationship and can find inverses
- **Function Composition:** You're comfortable with $f(g(x))$ notation and nested operations
- **Equation Solving:** You can solve various types of equations systematically
- **Desmos Mastery:** You're fluent in exploring mathematical relationships visually

If any of these feel uncertain, a quick review of Chapter 5 will help!

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll think of exponential and logarithmic functions as **the ultimate growth and measurement experts** who can model everything from bacterial colonies to earthquake intensity to compound interest.

The Story: You'll discover that exponential functions capture explosive growth patterns you see everywhere in nature and finance, while logarithmic functions provide the perfect measuring tools for phenomena that span huge ranges.

The Skills You'll Have:

- Model exponential growth and decay in real-world situations
- Understand logarithms as the inverse of exponentials
- Use properties of logarithms to solve complex equations
- Apply exponential and logarithmic models to business, science, and everyday life
- Recognize when exponential or logarithmic thinking is needed

The Confidence Moment: When you see problems involving compound interest, population growth, or pH levels, your brain will think "this is exponential/logarithmic territory" instead of "impossible word problem."

The Bridge: Chapter 7 will show you how multiple equations work together as systems, building on the equation-solving skills you develop here with exponentials and logarithms.

6.1 Exponential Functions (The Growth Champions)

READ THIS FIRST

You accidentally leave a piece of bread on your kitchen counter.
Day 1: You notice a tiny speck of mold. Just one small dot.
Day 2: That one speck has become two specks. Hmm.
Day 3: Four specks. This is getting interesting.
Day 4: Eight specks. Wait, this is accelerating.
Day 5: Sixteen specks. Okay, this is getting out of hand.
Day 6: Thirty-two specks. YIKES.
Day 7: Sixty-four specks covering a noticeable area.
What's happening here? The mold isn't growing by adding the same amount each day (that would be linear growth: 1, 2, 3, 4, 5...).
Instead, it's **doubling** each day. It's growing by a constant **factor** rather than a constant **amount**.
This is exponential growth: 1, 2, 4, 8, 16, 32, 64...
If we let t represent the day and $M(t)$ represent the number of mold specks, then:

$$M(t) = 1 \cdot 2^t$$

This pattern—where the variable is in the exponent—shows up everywhere: - Bacterial colonies doubling every hour - Money growing with compound interest - Viral videos spreading on social media - Radioactive materials decaying over time - Population growth in cities Exponential functions are nature's way of saying "things can change REALLY fast when they build on themselves."

LET'S TALK ABOUT IT

Think about situations where you've seen explosive growth or rapid change:

- Social media — a post going viral, follower counts exploding
- Technology — computer processing power, internet speeds
- Health — how quickly you recover (or don't) from illness
- Learning — skill improvement when you practice consistently
- Finance — how debt can spiral, or how savings can compound

What patterns do you notice? How is this different from steady, linear growth?

NOW WE NAME IT

An **Exponential Function** has the form:

$$f(x) = a \cdot b^x$$

where $a > 0$ is the initial value and $b > 0, b \neq 1$ is the base.

Key Components:

- **Base b:** Determines the growth/decay factor
- **Initial value a:** The y-intercept, what you start with
- **Variable x:** Appears in the exponent (this is what makes it exponential!)

Types of Exponential Behavior:

Exponential Growth ($b > 1$):

- Function values increase rapidly
- Each unit increase in x multiplies the output by b
- Example: $f(x) = 2^x$ (doubles each step)

Exponential Decay ($0 < b < 1$):

- Function values decrease rapidly toward zero
- Each unit increase in x multiplies the output by b (a fraction)
- Example: $f(x) = \left(\frac{1}{2}\right)^x$ (halves each step)

Special Base: $e \approx 2.718$

- The "natural" exponential function: $f(x) = e^x$
- Shows up constantly in calculus, statistics, and natural phenomena
- Think of e as the "perfect" growth rate

Key Properties:

- Domain: All real numbers
- Range: $(0, +\infty)$ (always positive, never touches x-axis)
- y-intercept: $(0, a)$
- No x-intercepts
- Either always increasing ($b > 1$) or always decreasing ($0 < b < 1$)

Real-World Applications:

- Population growth: $P(t) = P_0 \cdot r^t$
- Compound interest: $A(t) = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$
- Radioactive decay: $N(t) = N_0 \cdot \left(\frac{1}{2}\right)^{t/h}$
- Bacterial growth: $B(t) = B_0 \cdot 2^{t/d}$

WATCH IT WORK

Example 1: Model bacterial growth
A bacterial culture starts with 500 bacteria and doubles every 3 hours.
Set up the function:

- Initial amount: $a = 500$
- Doubles every 3 hours, so after 3 hours we have $500 \cdot 2$
- After 6 hours: $500 \cdot 2^2$
- After t hours: $B(t) = 500 \cdot 2^{t/3}$

Questions and answers:
Q: How many bacteria after 9 hours?

$$B(9) = 500 \cdot 2^{9/3} = 500 \cdot 2^3 = 500 \cdot 8 = 4000 \text{ bacteria}$$

Q: When will there be 16,000 bacteria?

$$16000 = 500 \cdot 2^{t/3}$$

$$32 = 2^{t/3}$$
$$2^5 = 2^{t/3}$$
$$5 = \frac{t}{3}$$
$$t = 15 \text{ hours}$$

Desmos Exploration:

1. Graph `y = 500 * 2^(x/3)`
2. Notice the characteristic exponential curve
3. Check the points: $(0, 500)$, $(3, 1000)$, $(6, 2000)$, $(9, 4000)$
4. See how rapidly it grows!

Example 2: Compound interest magic
You invest \$1,000 at 5% annual interest, compounded annually.
The compound interest formula:

$$A(t) = P \cdot (1 + r)^t$$

where P = principal, r = interest rate, t = time in years
Your investment:

$$A(t) = 1000 \cdot (1.05)^t$$

Calculate growth:

- After 1 year: $A(1) = 1000 \cdot 1.05 = \$1,050$
- After 10 years: $A(10) = 1000 \cdot (1.05)^{10} \approx \$1,629$
- After 20 years: $A(20) = 1000 \cdot (1.05)^{20} \approx \$2,653$
- After 50 years: $A(50) = 1000 \cdot (1.05)^{50} \approx \$11,467$

The magic: Your money doesn't just grow—it grows faster and faster because you earn

YOUR TURN

1. **Exponential Recognition:** Identify whether each represents exponential growth, decay, or neither:
   a) A population that increases by 200 people per year
   b) A population that increases by 3% per year
   c) Money that earns $50 per month in interest
   d) Medicine that loses half its effectiveness every 6 hours
2. **Function Building:** Write exponential functions for these scenarios:
   a) $2,000 invested at 4% annual growth
   b) A virus that triples every day, starting with 10 cases
   c) A substance with initial mass 80g that loses 20% per hour
3. **Desmos Exploration:**
   a) Graph `y = 2^x` and `y = 3^x` on the same axes
   b) Which grows faster? Why?
   c) Add `y = (0.5)^x` — how is this different?
   d) Find where $2^x = 8$ using your graph
4. **Real-World Application:** A social media post gets 5 likes initially and the likes triple every hour.
   a) Write the function $L(t)$ for likes after t hours
   b) How many likes after 4 hours?
   c) When will the post reach 1,215 likes?
   d) Graph this in Desmos and describe the growth pattern

CHECK YOUR VOICE

What did your brain say when exploring exponential functions?
If it said *"The numbers get big really fast!"* — that's the power of exponential growth! Small changes in the base or time create dramatic differences in outcomes.
If it said *"This explains why compound interest is so powerful"* — exactly! Einstein supposedly called compound interest "the eighth wonder of the world." Now you see the mathematics behind that magic.
If it said *"I can see this pattern in viral videos and population growth"* — you're recognizing exponential patterns everywhere! This mathematical model explains countless real-world phenomena.
If it said *"Decay functions make sense for things like medicine"* — great connection! Exponential decay models how drugs leave your system, how radioactive materials break down, how heat dissipates.

6.2 Logarithmic Functions (The Measurement Masters)

READ THIS FIRST

You're a scientist studying earthquakes.
The 1994 Northridge earthquake in Los Angeles had an intensity of about 1,000,000,000,000,000,000 times more than the smallest detectable earthquake.
That's 10^{18} times more intense.
But when scientists talk about earthquakes, they don't say "this was a 1,000,000,000,000,000,000 intensity earthquake." That's impossible to work with.
Instead, they use the Richter scale, and they say "this was a magnitude 6.7 earthquake."
How do they get from 10^{18} to 6.7?
They use a **logarithm**.
A logarithm answers the question: "To what power must I raise my base to get this number?"
So $\log_{10}(10^{18}) = 18$ because you need to raise 10 to the 18th power to get 10^{18}.
But the Richter scale is adjusted, so a magnitude 6.7 earthquake corresponds to intensity $10^{6.7}$.
Logarithms are the mathematical tool for taking huge ranges of numbers and compressing them into manageable scales.
Think about other examples: - pH scale (acid/base strength): ranges from 0 to 14 instead of 10^{-14} to 10^0 - Decibels (sound intensity): 0 to 120 instead of 1 to 10^{12} - Musical scales: octaves instead of frequency ratios
Logarithms are the inverse function of exponentials. If exponentials make things grow explosively, logarithms tame that explosion into something humans can work with.

LET'S TALK ABOUT IT

Think about measurement systems that compress huge ranges into smaller, manageable numbers:

- Star brightness — astronomers don't use raw brightness numbers
- Computer file sizes — KB, MB, GB, TB (each is about 1,000 times bigger)
- Economic scales — comparing personal income to national GDP
- Time scales — seconds vs. geological eras vs. cosmic time

Why do we need compressed scales? What makes some ranges of numbers hard to work with directly?

NOW WE NAME IT

A **Logarithmic Function** is the inverse of an exponential function.
Definition: $y = \log_b(x)$ means $b^y = x$
In words: "$\log_b(x)$ is the power you raise b to in order to get x"
Key Relationship with Exponentials: If $f(x) = b^x$, then $f^{-1}(x) = \log_b(x)$
This means:

- $b^{\log_b(x)} = x$ (for $x > 0$)
- $\log_b(b^x) = x$ (for all real x)

Common Logarithm Bases:

- **Base 10 (Common):** $\log(x) = \log_{10}(x)$ — used in science, engineering
- **Base e (Natural):** $\ln(x) = \log_e(x)$ — used in calculus, natural phenomena
- **Base 2:** $\log_2(x)$ — used in computer science, information theory

Key Properties:

- Domain: $(0, +\infty)$ — only positive numbers have real logarithms
- Range: All real numbers
- x-intercept: $(1, 0)$ because $\log_b(1) = 0$ (any base to power 0 equals 1)
- No y-intercept (function undefined at $x = 0$)
- Always increasing (if $b > 1$)
- Graph is reflection of $y = b^x$ across line $y = x$

Logarithm Properties:

1. **Product Rule:** $\log_b(xy) = \log_b(x) + \log_b(y)$
2. **Quotient Rule:** $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
3. **Power Rule:** $\log_b(x^n) = n \cdot \log_b(x)$
4. **Change of Base:** $\log_b(x) = \frac{\log(x)}{\log(b)}$

Real-World Applications:

- Earthquake intensity (Richter scale)
- Sound intensity (decibels)
- pH scale (acid/base strength)
- Information storage (bits and bytes)
- Economic modeling (income distributions)

WATCH IT WORK

Example 1: Understanding logarithms through exponentials
Let's build the connection between exponentials and logarithms:
If $2^3 = 8$, *then* $\log_2(8) = 3$ *If* $10^2 = 100$, *then* $\log_{10}(100) = 2$ *If* $5^0 = 1$, *then* $\log_5(1) = 0$
Calculate these logarithms by thinking exponentially:
Q: What is $\log_2(16)$*?* Think: "What power of 2 gives me 16?" $2^4 = 16$, so $\log_2(16) = 4$
Q: What is $\log_{10}(1000)$*?* Think: "What power of 10 gives me 1000?" $10^3 = 1000$, so $\log_{10}(1000) = 3$
Q: What is $\log_3\left(\frac{1}{9}\right)$*?* Think: "What power of 3 gives me $\frac{1}{9}$?" $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$, so $\log_3\left(\frac{1}{9}\right) = -2$
Desmos Investigation:

1. Graph `y = 2^x` and `y = log_2(x)` on the same axes
2. Add the line `y = x`
3. See how they're reflections of each other!
4. Check: Does the point $(3, 8)$ on $y = 2^x$ correspond to $(8, 3)$ on $y = \log_2(x)$?

Example 2: Using logarithm properties
Simplify using logarithm properties:
Q: Simplify $\log_3(9 \cdot 27)$
Method 1 — Direct calculation: $9 \cdot 27 = 243$, and $3^5 = 243$, so $\log_3(243) = 5$
Method 2 — Using product property:

$$\log_3(9 \cdot 27) = \log_3(9) + \log_3(27) \tag{6.1}$$
$$= \log_3(3^2) + \log_3(3^3) \tag{6.2}$$
$$= 2 + 3 = 5 \tag{6.3}$$

Q: Simplify $\log_2\left(\frac{32}{4}\right)$
Using quotient property:

$$\log_2\left(\frac{32}{4}\right) = \log_2(32) - \log_2(4) \tag{6.4}$$
$$= \log_2(2^5) - \log_2(2^2) \tag{6.5}$$
$$= 5 - 2 = 3 \tag{6.6}$$

Check: $\frac{32}{4} = 8 = 2^3$, so $\log_2(8) = 3$
Q: Simplify $\log_5(125^2)$
Using power property:

$$\log_5(125^2) = 2 \cdot \log_5(125) \tag{6.7}$$
$$= 2 \cdot \log_5(5^3) \tag{6.8}$$
$$= 2 \cdot 3 = 6 \tag{6.9}$$

Check: $125^2 = (5^3)^2 = 5^6$, so $\log_5(5^6) = 6$

Example 3: Real-world application — pH scale
The pH scale measures acidity using $\text{pH} = -\log_{10}[\text{H}^+]$
where $[\text{H}^+]$ is the hydrogen ion concentration.
Calculate pH values:
Q: If $[H^+] = 10^{-7}$*, what's the pH?*

YOUR TURN

1. **Logarithm Evaluation:** Calculate these logarithms by thinking exponentially:
   a) $\log_5(25)$
   b) $\log_2(64)$
   c) $\log_{10}(0.01)$
   d) $\log_4(1)$
2. **Properties Practice:** Use logarithm properties to simplify:
   a) $\log_7(7^4)$
   b) $\log_3(27) + \log_3(9)$
   c) $\log_2(16) - \log_2(4)$
   d) $\log_5(25^3)$
3. **Desmos Investigation:**
   a) Graph `y = log(x)` and `y = 10^x` together
   b) Add the line `y = x` — see the inverse relationship!
   c) Find where $\log(x) = 2$ using your graph
   d) Compare $\log(x)$ growth to linear growth — which is slower?
4. **Richter Scale Application:** The Richter scale uses $M = \log_{10}(I/I_0)$ where I is intensity.
   a) If one earthquake is 1,000 times more intense than another, how much higher is its Richter magnitude?
   b) An earthquake measures 7.2 on the Richter scale. Another measures 5.2. How many times more intense is the first?
   c) Why is the logarithmic scale useful for earthquakes?

CHECK YOUR VOICE

What did your brain say when working with logarithmic functions?
If it said *"Logarithms are just the reverse of exponentials"* — perfect! You've grasped the fundamental inverse relationship that makes logarithms work.
If it said *"The properties make complex calculations simpler"* — exactly! Logarithms were invented to turn multiplication into addition, making calculations manageable before computers.
If it said *"Now I understand why we need compressed scales"* — mathematical maturity! You see how logarithms solve the practical problem of working with huge number ranges.
If it said *"This explains pH and earthquakes and sound"* — brilliant connections! You're recognizing that logarithmic scaling appears everywhere in science because many natural phenomena span enormous ranges.

6.3 Solving Exponential and Logarithmic Equations

READ THIS FIRST

You're trying to figure out when your investment will double.
You invested \$5,000 at 6% annual interest, compounded annually. The formula is:

$$A(t) = 5000(1.06)^t$$

You want to know: when will $A(t) = 10,000$?
So you need to solve: $10,000 = 5000(1.06)^t$
Simplifying: $2 = (1.06)^t$
Now you're stuck. How do you solve for t when it's in the exponent?
This is where logarithms become your problem-solving superhero.
Since logarithms are the inverse of exponentials, they can "bring down" the exponent:

$$2 = (1.06)^t$$
$$\log(2) = \log((1.06)^t)$$
$$\log(2) = t \cdot \log(1.06)$$
$$t = \frac{\log(2)}{\log(1.06)}$$

Using a calculator: $t \approx \frac{0.301}{0.025} \approx 12.0$ years
The beautiful thing? This same technique works for any exponential equation. Logarithms unlock the variable trapped in the exponent.
And if you have a logarithmic equation? Use the inverse relationship! Convert to exponential form and solve.
These solving techniques let you answer real questions: When will the population reach a certain size? How long until a medicine is out of your system? At what pH is a solution dangerous?

LET'S TALK ABOUT IT

Think about situations where you need to solve for time or find unknown exponents:

- Financial planning — how long to reach savings goals?
- Medical dosing — when will drug concentration be safe?
- Environmental science — how long for pollutants to break down?
- Technology — when will computer processing reach certain speeds?
- Population studies — when will cities reach capacity?

What makes these "when will..." questions different from simpler algebra problems?

NOW WE NAME IT

Solving Exponential Equations

Strategy 1: Same Base Method When you can express both sides with the same base:

$$3^{x+1} = 9^{x-2}$$

$$3^{x+1} = (3^2)^{x-2}$$
$$3^{x+1} = 3^{2(x-2)}$$
$$x + 1 = 2(x - 2) = 2x - 4$$
$$x = 5$$

Strategy 2: Logarithm Method When bases are different or involve e:

$$5^x = 20$$

$$\log(5^x) = \log(20)$$
$$x \cdot \log(5) = \log(20)$$
$$x = \frac{\log(20)}{\log(5)}$$

Strategy 3: Natural Log for Base e

$$e^{2x-1} = 50$$

$$\ln(e^{2x-1}) = \ln(50)$$
$$2x - 1 = \ln(50)$$
$$x = \frac{\ln(50) + 1}{2}$$

Solving Logarithmic Equations

Strategy 1: Convert to Exponential

$$\log_3(x + 5) = 4$$

$$3^4 = x + 5$$
$$81 = x + 5$$
$$x = 76$$

Strategy 2: Use Logarithm Properties

$$\log_2(x) + \log_2(x - 6) = 4$$

$$\log_2(x(x - 6)) = 4$$
$$x(x - 6) = 2^4 = 16$$
$$x^2 - 6x - 16 = 0$$
$$(x - 8)(x + 2) = 0$$
$$x = 8 \text{ or } x = -2$$

Check: $x = -2$ gives negative argument, so $x = 8$ only.

Strategy 3: Isolate then Convert

WATCH IT WORK

Example 1: Investment doubling time

Find when $1,000 invested at 8% annual interest doubles.

Set up the equation:

$$2000 = 1000(1.08)^t$$

$$2 = (1.08)^t$$

Solve using logarithms:

$$\log(2) = \log((1.08)^t)$$

$$\log(2) = t \cdot \log(1.08)$$

$$t = \frac{\log(2)}{\log(1.08)} = \frac{0.3010}{0.0334} \approx 9.01 \text{ years}$$

Interpretation: The "Rule of 72" says investments double in about $\frac{72}{8} = 9$ years at 8%. Our calculation confirms this!

Desmos Verification: Graph `y = 1000(1.08)^x` and `y = 2000`, find intersection.

Example 2: Radioactive decay timing

Carbon-14 has half-life 5,730 years. How long until only 10% of original amount remains?

Set up the decay function:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{t/5730}$$

Find when $N(t) = 0.1N_0$:

$$0.1N_0 = N_0 \left(\frac{1}{2}\right)^{t/5730}$$

$$0.1 = \left(\frac{1}{2}\right)^{t/5730}$$

Solve using logarithms:

$$\log(0.1) = \log\left(\left(\frac{1}{2}\right)^{t/5730}\right)$$

$$\log(0.1) = \frac{t}{5730} \cdot \log\left(\frac{1}{2}\right)$$

$$\frac{t}{5730} = \frac{\log(0.1)}{\log(0.5)}$$

$$t = 5730 \cdot \frac{-1}{-0.301} \approx 19{,}035 \text{ years}$$

Check: After about 3.3 half-lives, we'd have $\left(\frac{1}{2}\right)^{3.3} \approx 0.1$

Example 3: Logarithmic equation with properties

Solve: $\log_3(x+1) + \log_3(x-2) = 2$

Use product property:

$$\log_3((x+1)(x-2)) = 2$$

Convert to exponential:

$$(x+1)(x-2) = 3^2 = 9$$

Expand and solve:

$$x^2 - 2x + x - 2 = 9$$

YOUR TURN

1. **Exponential Equations:** Solve for x:
   a) $2^x = 32$
   b) $3^{x-1} = 27$
   c) $5^x = 200$ (use calculator for decimal approximation)
   d) $e^{2x} = 50$
2. **Logarithmic Equations:** Solve and check your solutions:
   a) $\log_4(x) = 3$
   b) $\log(x) + \log(x - 9) = 1$
   c) $\ln(x + 1) = 2$
   d) $2\log_3(x) - 1 = 3$
3. **Real-World Problems:**
   a) A bacteria culture triples every 4 hours. If it starts with 100 bacteria, when will it reach 8,100 bacteria?
   b) You invest \$3,000 at 5.5% annual interest. When will your investment reach \$5,000?
   c) A medication decays so that 75% remains each hour. When will only 10% of the original dose remain?
4. **Desmos Problem Solving:**
   a) Graph `y = 2^x` and `y = 10` to solve $2^x = 10$
   b) Use your graph to estimate the solution, then calculate exactly
   c) Verify using the logarithm method

CHECK YOUR VOICE

What did your brain say when solving exponential and logarithmic equations?
If it said *"Taking the log of both sides unlocks the exponent"* — perfect insight! You understand the key strategy for solving exponential equations.
If it said *"I need to be careful about domain restrictions"* — excellent mathematical caution! Logarithms of negative numbers aren't real, so checking solutions is crucial.
If it said *"These solve real timing questions"* — exactly! Most "when will..." questions in science, finance, and everyday life lead to exponential or logarithmic equations.
If it said *"Converting between forms helps me see the solution"* — mathematical flexibility! Moving between exponential and logarithmic forms gives you multiple approaches to the same problem.

YOUR TURN - EXTENDED PRACTICE

Chapter 6: Mastering Exponential & Logarithmic Functions

Growth and Measurement Mastery

1. Function Family Recognition
Classify each scenario and write the appropriate function:

a) A population grows by 500 people per year (current: 10,000)

b) A population grows by 5% per year (current: 10,000)

c) Sound intensity on decibel scale: $d = 10\log\left(\frac{I}{I_0}\right)$

d) Radioactive decay where 15% is lost each year

e) Investment at 7% annual compound interest

For each exponential/logarithmic scenario, identify: initial value, base/rate, and whether it represents growth or decay.

2. Exponential Applications Championship

a) **Viral Social Media:** A video has 50 views and triples every 2 hours.

- Write the function $V(t)$ for views after t hours
- How many views after 8 hours?
- When will it reach 1.2 million views?
- Graph in Desmos and describe the growth pattern

b) **Compound Interest Comparison:** You have $10,000 to invest for 20 years.

- Option A: 6% annually
- Option B: 5.8% compounded monthly
- Calculate final amounts for both options
- Which is better? By how much?

c) **Medicine Decay:** A drug has half-life of 8 hours. Initial dose: 400mg.

- Write the decay function
- How much remains after 24 hours?
- When will only 25mg remain?
- Why is understanding decay important for dosing?

3. Logarithmic Scale Mastery

a) **pH Scale Applications:**

- Coffee has $[H^+] = 10^{-5}$. Calculate its pH.
- Soap has pH = 10. Find its $[H^+]$ concentration.
- Which is more acidic: pH 3 or pH 5? By what factor?

ANSWER KEY - CHAPTER 6

Section 1 - Your Turn: Exponential Functions

1. **Exponential Recognition:**
   a) Neither (linear growth - adds constant amount)
   b) Exponential growth (multiplies by constant factor 1.03)
   c) Neither (linear growth - adds constant amount)
   d) Exponential decay (loses half effectiveness every 6 hours)

2. **Function Building:**
   a) $A(t) = 2000(1.04)^t$
   b) $V(t) = 10 \cdot 3^t$
   c) $M(t) = 80(0.8)^t$ (loses 20%, so 80% remains)

3. **Desmos Exploration:**
   a) Both pass through $(0, 1)$, but 3^x grows faster than 2^x
   b) 3^x grows faster because base 3 ¿ base 2
   c) $(0.5)^x$ decreases (decay) while others increase (growth)
   d) $2^x = 8$ when $x = 3$ (since $2^3 = 8$)

4. **Social Media Application:**
   a) $L(t) = 5 \cdot 3^t$
   b) After 4 hours: $L(4) = 5 \cdot 3^4 = 5 \cdot 81 = 405$ likes
   c) For 1,215 likes: $1215 = 5 \cdot 3^t \Rightarrow 243 = 3^t \Rightarrow t = 5$ hours
   d) Exponential growth - rapid acceleration, characteristic J-curve

Section 2 - Your Turn: Logarithmic Functions

1. **Logarithm Evaluation:**
   a) $\log_5(25) = 2$ (since $5^2 = 25$)
   b) $\log_2(64) = 6$ (since $2^6 = 64$)
   c) $\log_{10}(0.01) = -2$ (since $10^{-2} = 0.01$)
   d) $\log_4(1) = 0$ (since $4^0 = 1$)
2. **Properties Practice:**
   a) $\log_7(7^4) = 4$
   b) $\log_3(27) + \log_3(9) = \log_3(27 \cdot 9) = \log_3(243) = \log_3(3^5) = 5$
   c) $\log_2(16) - \log_2(4) = \log_2(16/4) = \log_2(4) = 2$
   d) $\log_5(25^3) = 3\log_5(25) = 3 \cdot 2 = 6$
3. **Desmos Investigation:**
   a) Graphs are reflections across $y = x$ (inverse functions)
   b) $\log(x) = 2$ when $x = 100$ (since $10^2 = 100$)
   c) Logarithmic growth is much slower - approaches infinity gradually
4. **Richter Scale:**
   a) $\log_{10}(1000) = 3$, so 3 units higher on Richter scale
   b) Intensity ratio: $10^{7.2-5.2} = 10^2 = 100$ times more intense
   c) Logarithmic scale compresses huge intensity ranges into manageable numbers

Section 3 - Your Turn: Solving Equations

1. **Exponential Equations:**
   a) $2^x = 32 = 2^5$, so $x = 5$
   b) $3^{x-1} = 27 = 3^3$, so $x - 1 = 3$, thus $x = 4$
   c) $5^x = 200$, so $x = \frac{\log(200)}{\log(5)} \approx 3.29$
   d) $e^{2x} = 50$, so $2x = \ln(50)$, thus $x = \frac{\ln(50)}{2} \approx 1.96$
2. **Logarithmic Equations:**
   a) $\log_4(x) = 3$, so $x = 4^3 = 64$
   b) $\log(x) + \log(x - 9) = 1$, so $\log(x(x - 9)) = 1$, thus $x(x - 9) = 10$ $x^2 - 9x - 10 = 0 \Rightarrow (x - 10)(x + 1) = 0$, so $x = 10$ (check: $x > 9$ required)
   c) $\ln(x + 1) = 2$, so $x + 1 = e^2$, thus $x = e^2 - 1 \approx 6.39$
   d) $2\log_3(x) - 1 = 3$, so $2\log_3(x) = 4$, thus $\log_3(x) = 2$, so $x = 3^2 = 9$
3. **Real-World Problems:**
   a) $8100 = 100 \cdot 3^{t/4}$, so $81 = 3^{t/4}$, thus $3^4 = 3^{t/4}$, so $t = 16$ hours
   b) $5000 = 3000(1.055)^t$, so $\frac{5}{3} = (1.055)^t$, thus $t = \frac{\log(5/3)}{\log(1.055)} \approx 9.51$ years
   c) $0.1A_0 = A_0(0.75)^t$, so $0.1 = (0.75)^t$, thus $t = \frac{\log(0.1)}{\log(0.75)} \approx 8.00$ hours
4. **Desmos Problem Solving:**
   a) Intersection occurs at approximately $x \approx 3.32$
   b) Exact: $x = \frac{\log(10)}{\log(2)} = \frac{1}{\log(2)} \approx 3.32$
   c) Both methods give the same result!

Problem 1: Function Family Recognition

a) Linear growth: $P(t) = 10,000 + 500t$

b) Exponential growth: $P(t) = 10,000(1.05)^t$, initial = 10,000, rate = 5%, growth

c) Logarithmic scale (measurement function)

d) Exponential decay: $A(t) = A_0(0.85)^t$, rate = 15% loss, decay

e) Exponential growth: $A(t) = P(1.07)^t$, rate = 7%, growth

Problem 2: Exponential Applications Championship

a) **Viral Social Media:**

- $V(t) = 50 \cdot 3^{t/2}$
- After 8 hours: $V(8) = 50 \cdot 3^4 = 50 \cdot 81 = 4,050$ views
- For 1.2M views: $1,200,000 = 50 \cdot 3^{t/2}$, so $24,000 = 3^{t/2}$, $t = 2\log_3(24,000) \approx 18.6$ hours

b) **Compound Interest:**

- Option A: $A = 10,000(1.06)^{20} \approx \$32,071$
- Option B: $A = 10,000(1 + 0.058/12)^{240} \approx \$31,859$
- Option A is better by about \$212

c) **Medicine Decay:**

- $D(t) = 400 \cdot (0.5)^{t/8}$
- After 24 hours: $D(24) = 400 \cdot (0.5)^3 = 50\text{mg}$
- For 25mg: $25 = 400(0.5)^{t/8}$, $t = 8\log_{0.5}(25/400) = 32$ hours

Problem 3: Logarithmic Scale Mastery

a) **pH Scale:**

- Coffee pH: $-\log(10^{-5}) = 5$
- Soap: $[\text{H}^+] = 10^{-10}$
- pH 3 vs pH 5: pH 3 is $10^{5-3} = 100$ times more acidic

b) **Earthquakes:**

- Intensity ratio: $10^{7.8-6.2} = 10^{1.6} \approx 40$ times more intense
- Earthquake C magnitude: $M = 6.2 + \log_{10}(1000) = 6.2 + 3 = 9.2$

c) **Sound:**

- Concert vs conversation: $10^{(120-60)/10} = 10^6 = 1,000,000$ times more intense
- For 90 dB: intensity ratio $= 10^{90/10} = 10^9$

Problem 4: Equation Solving Tournament

a) $4^{x-1} = 16^{x+2} \Rightarrow (2^2)^{x-1} = (2^4)^{x+2} \Rightarrow 2(x-1) = 4(x+2) \Rightarrow x = -5$

b) $e^{3x} = 200 \Rightarrow 3x = \ln(200) \Rightarrow x = \frac{\ln(200)}{3} \approx 1.77$

c) $\log_5(x+3) + \log_5(x-1) = 1 \Rightarrow \log_5((x+3)(x-1)) = 1 \Rightarrow (x+3)(x-1) = 5$ $x^2 + 2x - 8 = 0 \Rightarrow x = 2$ (check: $x > 1$ required)

d) $3^{2x-1} = 7^{x+3} \Rightarrow (2x-1)\ln(3) = (x+3)\ln(7) \Rightarrow x = \frac{\ln(7)+\ln(3)}{\ln(3)-\ln(7)} \approx -6.13$

e) $\ln(x) - \ln(x-4) = 2 \Rightarrow \ln\left(\frac{x}{x-4}\right) = 2 \Rightarrow \frac{x}{x-4} = e^2$ $x = e^2(x-4) \Rightarrow x = \frac{4e^2}{e^2-1} \approx 4.31$

f) $2^x \cdot 5^x = 1000 \Rightarrow (10)^x = 1000 \Rightarrow x = 3$

Problem 5: Coffee Shop Chain Growth

a) $L_A(t) = 3 + 2t$, $L_B(t) = 3(1.4)^t$

b) After 10 years: $L_A(10) = 23$, $L_B(10) = 3(1.4)^{10} \approx 86$ locations

c) Plan B exceeds Plan A when: $3(1.4)^t > 3 + 2t$. Solving graphically: $t \approx 2.4$ years

d) Linear vs exponential growth patterns clearly visible in graphs

e) Investment after 15 years: Plan A costs \$6.6M, Plan B costs \$57.8M total

f) Recommendation depends on capital availability and market saturation

CHECK YOUR VOICE - AFTER CHAPTER 6

What did your brain say as you mastered exponential and logarithmic functions?
If it said *"Exponentials explain why small changes compound into huge differences"* — you've grasped one of the most important mathematical insights! This explains everything from wealth inequality to viral spread.
If it said *"Logarithms make huge number ranges manageable"* — exactly! You understand why scientists use logarithmic scales for earthquakes, sound, pH, and more.
If it said *"These functions are inverses working together"* — mathematical elegance! You see the beautiful relationship between growth processes and measurement tools.
If it said *"I can model real-world growth and decay now"* — practical mathematical power! You can analyze investments, population dynamics, environmental processes, and more.
If it said *"The equation-solving techniques unlock time-based questions"* — problem-solving mastery! You can answer "when will..." questions that matter in science, finance, and life.
You've completed the function trilogy: polynomials (teamwork), transformations (flexibility), and exponentials/logarithms (growth/measurement)!
Ready for Review Chapter B? We're about to synthesize these three powerful function families and see how they work together to model complex real-world relationships!

CHAPTER 6 COMPLETE!

You've now mastered the growth experts! You understand that exponential and logarithmic functions are:
Growth Champions: Exponential functions model explosive growth and decay processes that compound over time
Measurement Masters: Logarithmic functions compress huge ranges into manageable scales for scientific measurement
Inverse Partners: These function families work together as perfect mathematical opposites
Problem-Solving Tools: You can solve equations to answer real timing and intensity questions
Real-World Modelers: From compound interest to earthquake intensity to population dynamics
Most importantly: You see exponential and logarithmic thinking as essential tools for understanding how quantities change dramatically over time and how to measure phenomena across vast scales.
Ready for Review Chapter B? We're about to integrate polynomials, transformations, and exponentials/logarithms into a comprehensive mathematical toolkit for solving complex, multi-step real-world problems!
Your function mastery journey reaches its crescendo next!

REVIEW CHAPTER B: MASTERING THE FUNDAMENTALS

WHERE YOU'VE BEEN: THE ADVANCED FUNCTION JOURNEY

Chapter 4: The Cast Grows — Polynomials You discovered that polynomials are teams of linear functions working together, learned to factor them back into their linear components, and mastered polynomial behavior patterns and real-world applications.

Chapter 5: The Transformation Artists — Function Manipulation You learned that functions can be shifted, stretched, reflected, composed, and inverted like mathematical shapeshifters, giving you complete control over function relationships and enabling sophisticated modeling.

Chapter 6: The Growth Experts — Exponential & Logarithmic Functions You mastered exponential growth and decay patterns, understood logarithms as measurement tools and inverse functions, and learned to solve equations involving explosive growth or compressed scales.

WHERE YOU'RE GOING: INTEGRATED MASTERY

This review chapter will show you that these three function families work together as a complete mathematical toolkit for modeling complex, multi-faceted real-world relationships. You'll see how:

- Polynomial models can be transformed and combined with exponential behavior
- Real-world scenarios often require multiple function types working together
- Your function manipulation skills apply to all mathematical relationships
- Complex problems become manageable when you recognize which tools to use

The Confidence Moment: When you encounter complex mathematical modeling situations, you'll automatically think "what combination of polynomials, transformations, and exponentials/logarithms do I need?" instead of "this is too complicated for me."

6.4 The Mathematical Foundation: Function Family Reunion

READ THIS FIRST

Imagine you're the director of a nature documentary about ecosystems.
Your ecosystem has different mathematical "species":
The Polynomial Family — These are the collaborative species. They work in teams. A cubic polynomial $f(x) = x^3 - 4x^2 + x + 6$ represents three linear factors working together to create complex behavior patterns. They have predictable turning points, end behaviors, and can model things like projectile motion or profit optimization.
The Transformation Family — These are the adapters. They take any mathematical species and modify it for different environments. They can shift a function to a new location, stretch it to fit different scales, flip it to reverse direction, or compose functions to create multi-step processes.
The Growth/Measurement Family — These are the specialists. Exponentials model explosive population growth, viral spread, or compound interest. Logarithms measure phenomena across huge scales like earthquake intensity or sound levels.
But here's where it gets interesting: in real ecosystems, different species interact. They don't live in isolation.
The same is true for mathematical functions. Real-world problems often require multiple function families working together:
- A startup's growth might be polynomial initially (steady team building), then exponential (viral product adoption) - Population growth might be exponential early, then logarithmic as resources become limited - A projectile's height (polynomial) combined with air resistance (exponential decay)
Your job as a mathematical ecologist is to recognize which species (function types) are present in any situation and how they interact with each other.

LET'S TALK ABOUT IT

Think about complex systems you've observed:

- Social media growth — starts slow, goes viral, then levels off
- Learning a new skill — initial struggle, breakthrough period, mastery plateau
- City development — planned growth, boom periods, infrastructure limits
- Technology adoption — early adopters, mass market, saturation

How might different mathematical models apply to different phases of these processes?

CONCEPT REVIEW

Function Family Characteristics Summary
Polynomial Functions: The Team Players

- Form: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
- Personality: Smooth curves with predictable turning points and end behavior
- Strengths: Modeling optimization, projectile motion, area/volume problems
- Key Skills: Factoring, finding zeros, predicting behavior from degree and leading coefficient

Transformation Functions: The Shapeshifters

- Operations: Shifting, stretching, reflecting, composing, inverting
- Personality: Adapts any function to new situations and scales
- Strengths: Customizing models, multi-step processes, converting between relationships
- Key Skills: $f(x - h) + k$ transformations, $(f \circ g)(x)$ composition, finding $f^{-1}(x)$

Exponential/Logarithmic Functions: The Growth/Measurement Experts

- Forms: $f(x) = ab^x$ and $g(x) = \log_b(x)$
- Personality: Explosive growth/decay or compressed measurement scales
- Strengths: Compound interest, population dynamics, scientific measurement
- Key Skills: Solving exponential/logarithmic equations, applying growth/decay models

Function Integration Principles:

1. **Sequential Models:** Different functions for different time periods or phases
2. **Composite Models:** Functions of functions, like exponential growth with polynomial decay
3. **Transformed Models:** Standard functions adapted to specific contexts
4. **Comparative Models:** Multiple functions modeling different aspects of the same situation

6.5 Integrated Problem Solving: The Ultimate Toolkit

MASTERY CHECK

Problem-Solving Strategy Integration
When facing a complex mathematical modeling situation:
Step 1: Function Family Recognition

- Does it involve steady change? (Linear)
- Teams of factors interacting? (Polynomial)
- Explosive growth or decay? (Exponential)
- Huge range compression? (Logarithmic)
- Function modification needed? (Transformations)
- Multi-step processes? (Composition)
- Reversible relationships? (Inverse functions)

Step 2: Model Construction

- Identify variables and parameters
- Choose appropriate function family
- Apply necessary transformations
- Build equations or systems

Step 3: Solution and Analysis

- Use appropriate solving techniques
- Interpret results in context
- Check reasonableness of answers
- Consider domain and range restrictions

Step 4: Communication and Verification

- Explain methodology and reasoning
- Use Desmos for visual verification
- Consider alternative approaches
- Assess model limitations

WATCH IT WORK

Integrated Example 1: Tech Startup Growth Analysis
A tech startup's user base grows according to different phases:
Phase 1 (Months 0-6): Steady growth as founders recruit friends

$$U_1(t) = 50 + 25t \quad \text{(linear)}$$

Phase 2 (Months 6-18): Exponential growth as product goes viral

$$U_2(t) = 200 \cdot 2^{(t-6)/4} \quad \text{(exponential, shifted)}$$

Phase 3 (Months 18+): Growth slows due to market saturation

$$U_3(t) = 3200 + 500 \log_2(t - 16) \quad \text{(logarithmic, shifted)}$$

Questions and analysis:
Q: How many users at the end of each phase?

- End of Phase 1: $U_1(6) = 50 + 25(6) = 200$ users
- End of Phase 2: $U_2(18) = 200 \cdot 2^{(18-6)/4} = 200 \cdot 2^3 = 1600$ users
- After 24 months total: $U_3(24) = 3200 + 500 \log_2(24 - 16) = 3200 + 500 \log_2(8) = 3200 + 500(3) = 4700$ users

Q: In which phase is growth fastest? Phase 2 (exponential) shows the steepest growth rate.
Q: What mathematical concepts are integrated here?

- Function families (linear, exponential, logarithmic)
- Transformations (horizontal shifts for phase timing)
- Piecewise function modeling (different rules for different domains)
- Real-world application (business modeling)

Desmos Modeling: Graph all three phases as a piecewise function to see the complete growth story!

INTEGRATED PRACTICE

Multi-Concept Challenge Problems
Problem Set 1: Function Family Integration

1. **Environmental Science:** A pollutant concentration follows $C(t) = 100e^{-0.1t}$ (exponential decay), but seasonal effects add a polynomial variation: $S(t) = 2t^2 - 8t + 15$. The total concentration is $T(t) = C(t) + 0.1S(t)$.

   a) Calculate $T(5)$
   b) When is the concentration purely from decay (no seasonal effect)?
   c) Graph both components and their sum in Desmos

2. **Economics:** A company's profit model combines:

   - Base profit: $P(x) = -2x^2 + 80x - 200$ (polynomial)
   - Growth factor from marketing: $G(m) = 1.5^m$ (exponential)
   - Total profit: $T(x, m) = P(x) \cdot G(m)$

   a) Find the profit with $x = 15$ units and $m = 4$ marketing level
   b) What production level maximizes $P(x)$?
   c) How does increasing marketing level affect total profit?

3. **Population Dynamics:** A city's growth follows $P(t) = 50000(1.03)^t$, but housing capacity limits create the constraint $H(P) = P + 10000\log_{10}(P/40000)$.

   a) Find when population reaches 75,000
   b) Calculate housing capacity for that population
   c) What happens when $P(t) > H(P)$?

Problem Set 2: Transformation and Composition Mastery

1. **Temperature Conversion Chain:**

   - Fahrenheit to Celsius: $C(F) = \frac{5}{9}(F - 32)$
   - Celsius to Kelvin: $K(C) = C + 273.15$
   - Direct conversion: $(K \circ C)(F)$

   a) Find the composite function $(K \circ C)(F)$
   b) Convert 100°F directly to Kelvin
   c) Find the inverse function to convert Kelvin to Fahrenheit

2. **Investment Growth with Fees:** An investment grows according to $A(t) = 1000(1.08)^t$, but annual fees of \$50 are deducted, giving net value $N(t) = A(t) - 50t$.

   a) Graph both $A(t)$ and $N(t)$
   b) When does the fee impact become significant?
   c) Find when net value equals initial investment

3. **Function Design Challenge:** Create a function that:

6.6 Mastery Verification: The Complete Function Toolkit

MASTERY CHECK

Self-Assessment: Advanced Function Mastery
Rate your confidence (1-5 scale) in each area:
Polynomial Mastery:

- I can factor polynomials and find zeros ___
- I can predict polynomial behavior from degree and leading coefficient ___
- I can apply polynomial models to optimization problems ___
- I can graph polynomials and identify key features ___

Transformation Mastery:

- I can apply all transformation types (shift, stretch, reflect) ___
- I can compose functions and find domains ___
- I can find and verify inverse functions ___
- I can recognize when transformations are needed ___

Exponential/Logarithmic Mastery:

- I can model growth and decay situations ___
- I can solve exponential and logarithmic equations ___
- I can apply logarithmic scales (pH, Richter, decibels) ___
- I can explain the inverse relationship between exponentials and logs ___

Integration Mastery:

- I can recognize which function families apply to real situations ___
- I can combine multiple function types in one model ___
- I can use Desmos effectively for complex function analysis ___
- I can communicate mathematical reasoning clearly ___

Scoring:

- 16-20: Advanced Function Master — Ready for any mathematical modeling challenge!
- 12-15: Strong Function User — Solid foundation with some areas for growth
- 8-11: Developing Function Skills — Good progress, continue practicing integration
- 4-7: Beginning Function Learner — Focus on individual concepts before integration

CHECK YOUR VOICE

What does your brain say about your advanced function journey?
If it says *"I can see how these function families work together"* — you've achieved mathematical integration! You're thinking like a mathematical modeler.
If it says *"Real-world problems need multiple tools"* — exactly! Most important applications require combining different mathematical approaches.
If it says *"I can choose the right function type for the situation"* — mathematical maturity! You're developing the judgment to select appropriate tools.
If it says *"Complex problems feel manageable now"* — confidence transformation! You've built a complete toolkit and the skills to use it.
If it says *"I want to tackle more challenging applications"* — mathematical curiosity awakened! You're ready for advanced mathematical modeling.
If it says *"Sometimes I still feel overwhelmed"* — that's completely normal! Advanced integration takes practice. Your foundation is solid.
Remember: You've gone from basic function understanding to advanced mathematical modeling mastery. That's an incredible achievement that reflects both your mathematical ability and your growth mindset!

6.7 Looking Forward: The Bridge to Advanced Mathematics

CONCEPT REVIEW

Your Mathematical Journey Continues
With your advanced function toolkit complete, you're ready for:
Chapter 7: The Team Players (Systems of Equations & Matrices) You'll learn how multiple equations work together simultaneously, extending your function work to multi-variable situations.
Chapter 8: The Shape Storytellers (Conic Sections) You'll explore circles, ellipses, parabolas, and hyperbolas as geometric extensions of your function knowledge.
Chapter 9: The Pattern Masters (Sequences, Series & Mathematical Induction) You'll investigate discrete patterns and infinite processes that build on your exponential and logarithmic understanding.
Beyond College Algebra: Your function mastery provides the foundation for calculus, statistics, differential equations, and any mathematical modeling you encounter in your field.
Real-World Applications: You now have the tools to model complex relationships in business, science, technology, social sciences, and everyday life.
The Mathematical Confidence You've Built: More than specific techniques, you've developed mathematical thinking, problem-solving strategies, and the confidence to approach unfamiliar situations systematically.
Your Function Families Are Forever: Polynomials, transformations, exponentials, and logarithms will appear throughout your mathematical journey. You now have deep, intuitive understanding of these fundamental building blocks.

REVIEW CHAPTER B COMPLETE!

You have achieved **Advanced Function Mastery**! You can:
Recognize Function Families: Instantly identify which mathematical tools apply to any situation
Integrate Multiple Concepts: Combine polynomials, transformations, and exponentials/logarithms seamlessly
Model Complex Relationships: Build sophisticated mathematical models for real-world applications
Solve Advanced Problems: Use your complete toolkit to tackle multi-step, multi-concept challenges
Think Mathematically: Approach unfamiliar problems with confidence and systematic reasoning
Most importantly: You see advanced mathematics not as a collection of isolated rules, but as a powerful, integrated system for understanding and influencing the world around you.
Ready for the final chapters? You're equipped with the mathematical sophistication to tackle systems of equations, geometric relationships, and discrete mathematics with confidence!
Your mathematical transformation is nearly complete — and it's spectacular!

ANSWER KEY - REVIEW CHAPTER B

Problem Set 1: Function Family Integration

1. **Environmental Science:**

   a) $T(5) = 100e^{-0.1(5)} + 0.1(2(25) - 8(5) + 15) = 100e^{-0.5} + 0.1(25) = 60.65 + 2.5 = 63.15$

   b) Seasonal effect is zero when $S(t) = 0$: $2t^2 - 8t + 15 = 0$ Using quadratic formula: $t = \frac{8 \pm \sqrt{64-120}}{4}$ — no real solutions So seasonal effect is always present (minimum at $t = 2$)

   c) Desmos shows exponential decay with polynomial oscillation overlay

2. **Economics:**

   a) $T(15, 4) = P(15) \cdot G(4) = (-2(225) + 80(15) - 200) \cdot 1.5^4 = (-450 + 1200 - 200) \cdot 5.0625 = 550 \cdot 5.0625 = 2784.38$

   b) Maximum of $P(x) = -2x^2 + 80x - 200$: $x = -\frac{80}{2(-2)} = 20$ units

   c) Each marketing level multiplies profit by factor of 1.5

3. **Population Dynamics:**

   a) $75000 = 50000(1.03)^t$, so $1.5 = (1.03)^t$, thus $t = \frac{\ln(1.5)}{\ln(1.03)} \approx 13.7$ years

   b) $H(75000) = 75000 + 10000 \log_{10}(75000/40000) = 75000 + 10000 \log_{10}(1.875) \approx 77,735$

   c) When population exceeds housing capacity, overcrowding and infrastructure strain occur

Problem Set 2: Transformation and Composition Mastery

1. **Temperature Conversion:**
   a) $(K \circ C)(F) = K(C(F)) = \frac{5}{9}(F - 32) + 273.15 = \frac{5F}{9} - \frac{160}{9} + 273.15$
   b) $(K \circ C)(100) = \frac{5(100)}{9} - 17.78 + 273.15 = 310.93$ K
   c) From $K = \frac{5F}{9} + 255.37$: $F = \frac{9(K-255.37)}{5} = 1.8K - 459.67$

2. **Investment with Fees:**
   a) $A(t)$ grows exponentially, $N(t)$ grows more slowly due to linear fee deduction
   b) Fee impact becomes significant when linear term $50t$ approaches exponential growth rate
   c) $N(t) = 1000$: $1000(1.08)^t - 50t = 1000$, solve graphically: $t \approx 0$ or $t \approx 16.8$ years

3. **Function Design:** Starting with $f(x) = x^2$:
   - Shift right 3, up 2: $(x - 3)^2 + 2$
   - Reflect across x-axis: $-((x - 3)^2 + 2)$
   - Stretch vertically by 2: $-2((x - 3)^2 + 2) = -2(x - 3)^2 - 4$

   Final function: $g(x) = -2(x - 3)^2 - 4$

Problem Set 3: Advanced Integration

1. **Epidemiology Model:**
   a) $I(14) = 100 \cdot 3^{14/7} = 100 \cdot 9 = 900$ infections $(L \circ I)(14) = L(900) = \frac{10000(900)}{900+5000} = \frac{9{,}000{,}000}{5900} \approx 1525$ limited infections
   b) Limitations become significant when I approaches 5000 (around 3-4 weeks)
   c) Unlimited grows exponentially forever; limited asymptotes to 10,000

2. **Engineering Optimization:**
   a) $F(20, 30) = 2(400)+100(20)+50(1.01)^{30} = 800+2000+50(1.348) = 3467.4$ units
   b) Polynomial component dominates for normal loads; exponential becomes significant at high temperatures
   c) Minimum of $2L^2 + 100L$: $\frac{d}{dL}(2L^2 + 100L) = 4L + 100 = 0$, so $L = -25$ (not physically meaningful; no minimum for $L > 0$)

3. **Data Science Application:**
   a) At $t = 3$: $V(3) = 1000+200\sin(\pi/2) = 1200$, $R(1200) = 0.05\log_{10}(1200) \approx 0.154$
   $Rev(3) = 1200 \times 0.154 = 184.8$
   At $t = 9$: $V(9) = 1000 + 200\sin(3\pi/2) = 800$, $R(800) = 0.05\log_{10}(800) \approx 0.145$
   $Rev(9) = 800 \times 0.145 = 116$
   b) Logarithmic conversion makes sense because it represents diminishing returns — additional traffic becomes less effective at generating conversions
   c) Desmos shows sinusoidal traffic with logarithmic scaling creating revenue patterns

CHECK YOUR VOICE - AFTER COMPLETING ADVANCED FUNCTIONS

What does your mathematical voice say now?
If it says *"I can handle complex, multi-step mathematical modeling"* — you've achieved advanced mathematical confidence! You're ready for professional-level problem solving.
If it says *"I see how different mathematical tools work together"* — mathematical integration! You think like a mathematician, not just someone who knows math formulas.
If it says *"Real-world problems make more sense now"* — practical mathematical wisdom! You recognize that complex situations require sophisticated mathematical thinking.
If it says *"I'm curious about even more advanced mathematics"* — mathematical growth mindset! Your foundation is strong enough to support continued learning.
If it says *"Sometimes I still need to review individual concepts"* — that's perfect mathematical honesty! Mastery is about knowing when to revisit fundamentals.
You've completed an extraordinary mathematical journey: From basic functions to advanced mathematical modeling, from "I'm not a math person" to "I can tackle complex quantitative challenges with confidence."
Your mathematical voice has been transformed!

Chapter 7

THE TEAM PLAYERS

Systems of Equations & Matrices

SKILLS CHECK: What You Need From Previous Chapters

- **Linear Functions:** You're comfortable with equations like $y = mx + b$ and can find intersections
- **Equation Solving:** You can solve single equations using substitution, elimination, and graphing
- **Function Composition:** You understand how multiple mathematical processes work together
- **Coordinate Plane:** You can plot points, lines, and interpret graphs confidently
- **Desmos Fluency:** You're skilled at visual mathematical exploration and verification

If any of these feel uncertain, a quick review of earlier chapters will help!

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll think of systems of equations as **mathematical orchestras** where multiple equations work together simultaneously to solve complex, multi-constraint problems.

The Story: You'll discover that many real-world situations involve multiple relationships happening at the same time, and systems of equations provide the perfect tool for finding solutions that satisfy all constraints simultaneously.

The Skills You'll Have:

- Solve systems of equations using graphing, substitution, and elimination methods
- Use matrices to organize and solve complex systems efficiently
- Model real-world multi-constraint problems (business optimization, resource allocation)
- Recognize when problems require simultaneous equation thinking
- Apply matrix operations to streamline complex calculations

The Confidence Moment: When you see problems involving multiple unknowns and multiple constraints, your brain will think "this is a system of equations situation" instead of "too many variables to handle."

The Bridge: Chapter 8 will show you geometric curves (conics) that often involve systems of equations for finding intersections and analyzing relationships.

7.1 Systems of Equations (Mathematical Harmony)

READ THIS FIRST

You're planning a concert with your two best friends, Alex and Jordan.
You need to figure out ticket prices that work for everyone:
Alex says: "I want the total cost of 2 adult tickets and 3 student tickets to be \$50." Jordan says: "I need 1 adult ticket and 2 student tickets to cost \$22."
Now you have a puzzle. You need to find ticket prices that make **both** friends happy simultaneously.
Let's call the adult ticket price a and the student ticket price s.
Alex's requirement: $2a + 3s = 50$ Jordan's requirement: $a + 2s = 22$
You have two equations with two unknowns. This is a **system of equations**.
The beautiful thing? There's exactly one solution that satisfies both requirements at the same time.
Think about this geometrically: each equation represents a line on the coordinate plane. The solution is where these lines intersect — the one point that lies on both lines.
This is how the mathematical world handles complexity: when you have multiple constraints that must all be satisfied simultaneously, you create a system and find the point (or points) where all constraints are met.
Systems show up everywhere: - Business: balancing profit, costs, and production constraints - Engineering: designing structures that meet multiple safety and efficiency requirements - Economics: finding equilibrium where supply meets demand - Nutrition: meeting multiple dietary requirements with available foods - Scheduling: coordinating multiple people's availability and preferences
Systems of equations are mathematical teamwork at its finest.

LET'S TALK ABOUT IT

Think about situations in your life where you need to satisfy multiple requirements simultaneously:

- Planning a budget that covers all expenses within your income
- Choosing classes that meet graduation requirements and fit your schedule
- Recipe scaling that uses available ingredients while feeding the right number of people
- Travel planning that balances time, cost, and destinations
- Group projects where everyone's skills and availability must work together

How do you typically approach situations with multiple competing constraints?

NOW WE NAME IT

A **System of Equations** is a collection of two or more equations that must be satisfied simultaneously.

General Form (2 variables, 2 equations):

$$a_1x + b_1y = c_1 \tag{7.1}$$

$$a_2x + b_2y = c_2 \tag{7.2}$$

Solution Types:

- **One Solution:** Lines intersect at exactly one point (most common case)
- **No Solution:** Lines are parallel (inconsistent system)
- **Infinite Solutions:** Lines are the same (dependent system)

Solution Methods:

1. Graphing Method:

- Graph both equations on the same coordinate plane
- Find the intersection point(s)
- Best for visualization and understanding

2. Substitution Method:

- Solve one equation for one variable
- Substitute into the other equation
- Best when one equation is already solved for a variable

3. Elimination Method:

- Multiply equations to create opposite coefficients
- Add equations to eliminate one variable
- Best for equations in standard form

Real-World Applications:

- Business: Cost analysis, break-even points, resource allocation
- Science: Mixing problems, equilibrium conditions
- Economics: Supply and demand intersections
- Engineering: Multi-constraint optimization
- Social Sciences: Population dynamics, voting analysis

Key Insight: Systems represent situations where multiple relationships must hold true at the same time. The solution is the set of values that satisfies all equations simultaneously.

WATCH IT WORK

Example 1: Concert ticket pricing (from our opening story)
System:

$$2a + 3s = 50 \quad \text{(Alex's constraint)} \tag{7.3}$$
$$a + 2s = 22 \quad \text{(Jordan's constraint)} \tag{7.4}$$

Method 1: Substitution
From equation 2: $a = 22 - 2s$
Substitute into equation 1:

$$2(22 - 2s) + 3s = 50 \tag{7.5}$$
$$44 - 4s + 3s = 50 \tag{7.6}$$
$$44 - s = 50 \tag{7.7}$$
$$-s = 6 \tag{7.8}$$
$$s = -6 \tag{7.9}$$

$$2a + 3s = 35 \quad \text{(Alex's constraint)} \tag{7.10}$$
$$a + 2s = 20 \quad \text{(Jordan's constraint)} \tag{7.11}$$

From equation 2: $a = 20 - 2s$
Substitute:

$$2(20 - 2s) + 3s = 35 \tag{7.12}$$
$$40 - 4s + 3s = 35 \tag{7.13}$$
$$40 - s = 35 \tag{7.14}$$
$$s = 5 \tag{7.15}$$

Then: $a = 20 - 2(5) = 10$
Solution: Adult tickets \$10, Student tickets \$5
Check: $2(10) + 3(5) = 35$ and $10 + 2(5) = 20$
Desmos Verification:

1. Graph $y = \frac{35-2x}{3}$ (from first equation)
2. Graph $y = \frac{20-x}{2}$ (from second equation)
3. Find intersection point: $(10, 5)$

Example 2: Business break-even analysis
A company has two products with these relationships:

- Total production: $x + y = 100$ units
- Profit constraint: $3x + 2y = 250$ dollars

Find production levels for each product.
Using substitution:
From equation 1: $y = 100 - x$
Substitute into equation 2:

YOUR TURN

1. **Basic Systems Practice:** Solve using any method:

   a) $\begin{cases} x + y = 7 \\ 2x - y = 5 \end{cases}$

   b) $\begin{cases} 3x + 2y = 12 \\ x - y = 1 \end{cases}$

   c) $\begin{cases} 2x + 3y = 8 \\ 4x + 6y = 16 \end{cases}$

   d) $\begin{cases} x + 2y = 5 \\ 2x + 4y = 7 \end{cases}$

2. **Method Comparison:** For the system $\begin{cases} 2x + y = 8 \\ x - y = 1 \end{cases}$:

   a) Solve using substitution
   b) Solve using elimination
   c) Verify by graphing in Desmos
   d) Which method felt most natural to you?

3. **Real-World Applications:**

   a) **Concession Stand:** Hot dogs cost \$3 and sodas cost \$2. If 50 items were sold for \$130 total, how many of each were sold?
   b) **Age Problem:** Sarah is currently 3 times as old as her daughter. In 10 years, Sarah will be twice as old as her daughter. Find their current ages.
   c) **Investment:** \$5000 is invested in two accounts. One earns 4% and another earns 6% annual interest. If the total interest after one year is \$260, how much was invested in each account?

4. **Desmos Systems Lab:**

   a) Graph the system $\begin{cases} y = 2x + 1 \\ y = -x + 7 \end{cases}$
   b) Find the intersection point
   c) Try graphing $\begin{cases} y = 2x + 3 \\ y = 2x - 1 \end{cases}$ — what do you notice?
   d) Experiment with $\begin{cases} y = x + 2 \\ y = x + 2 \end{cases}$ — how is this different?

CHECK YOUR VOICE

What did your brain say when working with systems of equations?

If it said *"This is like solving a puzzle with multiple clues"* — perfect analogy! Each equation gives you a constraint, and the solution satisfies all constraints simultaneously.

If it said *"I prefer one method over the others"* — that's completely normal! Different people think differently. Substitution feels algebraic, elimination feels systematic, graphing feels visual.

If it said *"Real-world problems make more sense now"* — exactly! Most practical situations involve multiple constraints, so systems are everywhere in applications.

If it said *"Some systems have no solution or infinite solutions"* — excellent observation! Not every set of constraints can be satisfied, and sometimes constraints are redundant.

7.2 Matrix Methods (Organized Mathematical Power)

READ THIS FIRST

You're the logistics coordinator for a food bank.
Today you need to organize donations:
Location A needs: 20 cans of soup, 15 boxes of pasta, 10 loaves of bread Location B needs: 30 cans of soup, 25 boxes of pasta, 20 loaves of bread Location C needs: 10 cans of soup, 35 boxes of pasta, 15 loaves of bread
You have three different suppliers who can provide different combinations of items.
Writing this out as individual equations would be a nightmare:
$20s + 15p + 10b =$ Location A requirement $30s + 25p + 20b =$ Location B requirement $10s + 35p + 15b =$ Location C requirement
But what if you could organize all this information into a clean, rectangular array?

$$\begin{bmatrix} 20 & 15 & 10 \\ 30 & 25 & 20 \\ 10 & 35 & 15 \end{bmatrix} \begin{bmatrix} s \\ p \\ b \end{bmatrix} = \begin{bmatrix} \text{Total A} \\ \text{Total B} \\ \text{Total C} \end{bmatrix}$$

This is a **matrix**. It's a way to organize and manipulate large amounts of related information efficiently.
Matrices are like mathematical spreadsheets. They let you: - Store complex data in organized rectangular arrays - Perform operations on entire datasets at once - Solve systems with many variables efficiently - Transform geometric objects - Analyze networks and relationships
Once you learn to think in matrices, you can handle problems that would be overwhelming with individual equations.

LET'S TALK ABOUT IT

Think about situations where you organize information in rows and columns:

- Spreadsheets for budgets, schedules, or data tracking
- Sports statistics tables with players and different performance metrics
- Class schedules showing times, rooms, and subjects
- Recipe scaling tables showing ingredients for different serving sizes
- Comparison charts for products with multiple features

What advantages do organized tables have over lists of individual pieces of information?

NOW WE NAME IT

A **Matrix** is a rectangular array of numbers arranged in rows and columns.
Basic Notation: An $m \times n$ matrix has m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Common Matrix Types:

- **Square Matrix:** Same number of rows and columns ($n \times n$)
- **Row Vector:** Matrix with one row ($1 \times n$)
- **Column Vector:** Matrix with one column ($m \times 1$)
- **Zero Matrix:** All entries are zero
- **Identity Matrix:** Square matrix with 1's on diagonal, 0's elsewhere

Matrix Operations:
1. Addition/Subtraction: Add corresponding entries (matrices must be same size)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

2. Scalar Multiplication: Multiply every entry by the scalar

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

3. Matrix Multiplication: Row-by-column multiplication (number of columns in first = number of rows in second)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Solving Systems with Matrices:
Augmented Matrix Method: 1. Write system as augmented matrix $[A|B]$ 2. Use row operations to reach reduced row echelon form 3. Read solution from final matrix
Matrix Equation Method: 1. Write system as $AX = B$ 2. Find inverse matrix A^{-1} (if it exists) 3. Solution is $X = A^{-1}B$
Applications:

- Computer graphics and 3D transformations
- Economic modeling and input-output analysis
- Network analysis and social media algorithms
- Data science and machine learning
- Engineering and physics simulations

WATCH IT WORK

Example 1: Basic matrix operations

Given: $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix}$

Addition: $A + B$

$$A + B = \begin{bmatrix} 2+5 & 3+1 \\ 1+2 & 4+6 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 3 & 10 \end{bmatrix}$$

Scalar multiplication: $3A$

$$3A = \begin{bmatrix} 3(2) & 3(3) \\ 3(1) & 3(4) \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 3 & 12 \end{bmatrix}$$

Matrix multiplication: AB

$$AB = \begin{bmatrix} 2(5)+3(2) & 2(1)+3(6) \\ 1(5)+4(2) & 1(1)+4(6) \end{bmatrix} = \begin{bmatrix} 16 & 20 \\ 13 & 25 \end{bmatrix}$$

Note: Matrix multiplication is NOT commutative. $AB \neq BA$ in general.

Example 2: Solving a system using augmented matrix

System: $\begin{cases} 2x + 3y = 7 \\ x + 2y = 4 \end{cases}$

Step 1: Write augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & 2 & 4 \end{array}\right]$$

Step 2: Row operations to get leading 1's

$R_1 \leftrightarrow R_2$ (swap rows):

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 3 & 7 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -1 & -1 \end{array}\right]$$

$R_2 \to -R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$:

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Step 3: Read solution: $x = 2$, $y = 1$

Check: $2(2) + 3(1) = 7$ and $2 + 2(1) = 4$

Example 3: Real-world matrix application — nutrition planning

A nutritionist plans meals using three foods with different nutritional content:

Food	Protein (g)	Carbs (g)	Fat (g)
A	20	30	5
B	15	40	10
C	25	20	8

Goal: 60g protein, 100g carbs, 25g fat

Matrix setup:

$$\begin{bmatrix} 20 & 15 & 25 \\ \end{bmatrix} \begin{bmatrix} x \\ \end{bmatrix} \quad \begin{bmatrix} 60 \\ \end{bmatrix}$$

YOUR TURN

1. **Matrix Operations:** Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$:

   a) Calculate $A + B$
   b) Calculate $2A$
   c) Calculate AB
   d) Try to calculate BA — is it the same as AB?

2. **Augmented Matrix Practice:** Solve using augmented matrices:

   a) $\begin{cases} x + 2y = 5 \\ 3x - y = 4 \end{cases}$

   b) $\begin{cases} 2x + y = 8 \\ x + 3y = 9 \end{cases}$

3. **Real-World Matrix Modeling:** A company makes three products with these resource requirements:

Product	Labor (hrs)	Materials ($)	Machine Time (hrs)
X	2	10	1
Y	3	15	2
Z	1	5	1

   a) Write the resource matrix
   b) If they make 5 units of X, 3 units of Y, and 4 units of Z, calculate total resource usage
   c) Set up (don't solve) the system if they have 50 labor hours, $200 for materials, and 30 machine hours available

4. **Matrix Size and Operations:**

   a) Can you add a 2×3 matrix to a 3×2 matrix? Why or why not?
   b) Can you multiply a 2×3 matrix by a 3×2 matrix? What size is the result?
   c) Create two 2×2 matrices where $AB \neq BA$

CHECK YOUR VOICE

What did your brain say when learning about matrices?
If it said *"This is like organizing data in a spreadsheet"* — exactly! Matrices are mathematical spreadsheets that let you manipulate entire datasets efficiently.
If it said *"Matrix multiplication seems backwards"* — it takes practice! Remember: row times column, and the sizes have to match in the middle.
If it said *"This makes large systems more manageable"* — you're seeing the power of matrices! They turn overwhelming multi-variable problems into organized, systematic processes.
If it said *"I can see applications in technology and data science"* — excellent insight! Matrices are fundamental to computer graphics, machine learning, and data analysis.
If it said *"The row operations feel like puzzle solving"* — that's the systematic beauty of matrix methods! Each operation moves you closer to the solution.

7.3 Advanced Applications (Systems in the Real World)

READ THIS FIRST

You're the operations manager for a music festival.
You have three different types of stages to set up:
Main Stage: Requires 8 crew members, 12 hours of setup time, costs \$5000 Side Stage: Requires 5 crew members, 8 hours of setup time, costs \$3000 Acoustic Stage: Requires 3 crew members, 5 hours of setup time, costs \$2000
You have constraints: - 50 crew members available total - 80 hours of setup time available - \$35,000 budget - You want to maximize the number of stages to give artists more performance opportunities
This is no longer a simple "solve for x and y" situation. You have: - Multiple variables (number of each stage type) - Multiple constraints (crew, time, money) - An optimization goal (maximize total stages)
This is where systems of equations meet real-world complexity. You're not just finding where lines intersect — you're finding the best solution within a region of feasible options.
Real organizations face these multi-constraint optimization problems constantly: - Airlines scheduling flights with pilot availability, aircraft capacity, and fuel costs - Manufacturers balancing production volumes with material costs, labor constraints, and market demand - Hospitals allocating staff, equipment, and beds across different departments - Restaurants planning menus that balance nutrition requirements, cost constraints, and customer preferences
Mathematical systems provide the framework for making these complex decisions systematically rather than just guessing.

LET'S TALK ABOUT IT

Think about complex planning situations you've encountered:

- Planning a vacation with budget, time, and activity constraints
- Organizing a group event considering everyone's preferences, schedules, and contributions
- Choosing courses that meet requirements while balancing difficulty and interest
- Meal planning that meets nutritional goals within budget and time constraints
- Project management with multiple team members, deadlines, and resource limitations

How do you typically balance competing priorities and constraints in complex situations?

NOW WE NAME IT

Advanced Systems Applications

Linear Programming: Optimization problems with linear constraints and linear objective functions.

Standard form:

- **Objective function:** Maximize or minimize $z = c_1x_1 + c_2x_2 + \cdots$
- **Constraints:** $a_{i1}x_1 + a_{i2}x_2 + \cdots \leq b_i$ (or $\geq$, $=$)
- **Non-negativity:** $x_i \geq 0$

Three-Variable Systems:

$$a_1x + b_1y + c_1z = d_1 \tag{7.27}$$
$$a_2x + b_2y + c_2z = d_2 \tag{7.28}$$
$$a_3x + b_3y + c_3z = d_3 \tag{7.29}$$

Geometric interpretation: Three planes in 3D space

- One solution: Planes intersect at one point
- No solution: Planes don't share a common intersection
- Infinite solutions: Planes intersect along a line or are identical

Matrix Applications in Technology:

- **Computer Graphics:** Rotation, scaling, translation matrices
- **Data Science:** Principal component analysis, regression
- **Machine Learning:** Neural network weight matrices
- **Economics:** Input-output models for economic sectors
- **Engineering:** Structural analysis, electrical circuits

Network Applications:

- Social media connection matrices
- Transportation network optimization
- Supply chain management
- Internet routing protocols

Problem-Solving Strategy:

1. Identify variables and constraints clearly
2. Set up the system of equations or inequalities
3. Choose appropriate solution method

4. Solve systematically
5. Interpret results in context

WATCH IT WORK

Example 1: Three-variable business optimization
A bakery makes three products: breads, pastries, and cakes.
Resource constraints:

- Flour: 2 cups per bread, 1 cup per pastry, 3 cups per cake (24 cups available)
- Labor: 1 hour per bread, 2 hours per pastry, 3 hours per cake (18 hours available)
- Oven time: 30 min per bread, 20 min per pastry, 45 min per cake (300 min available)

Let b = breads, p = pastries, c = cakes
System:

$$2b + p + 3c = 24 \quad \text{(flour)} \tag{7.30}$$
$$b + 2p + 3c = 18 \quad \text{(labor)} \tag{7.31}$$
$$0.5b + 0.33p + 0.75c = 5 \quad \text{(oven time in hours)} \tag{7.32}$$

Simplifying equation 3: $30b + 20p + 45c = 300$ Dividing by 5: $6b + 4p + 9c = 60$
Augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 24 \\ 1 & 2 & 3 & 18 \\ 6 & 4 & 9 & 60 \end{array}\right]$$

After row operations:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution interpretation: The system has infinitely many solutions because the third constraint is redundant.
General solution: $b = 8 - c$, $p = 2 - c$, c is free
Practical constraints: $b \geq 0$, $p \geq 0$, $c \geq 0$ This gives us: $c \leq 2$
Possible production plans: - If $c = 0$: Make 8 breads, 2 pastries, 0 cakes - If $c = 1$: Make 7 breads, 1 pastry, 1 cake - If $c = 2$: Make 6 breads, 0 pastries, 2 cakes

Example 2: Investment portfolio optimization
An investor has $100,000 to allocate among three investments:

- Stocks (s): Expected return 8%, high risk
- Bonds (b): Expected return 4%, medium risk
- Savings (v): Expected return 2%, low risk

Goals:

- Invest entire amount: $s + b + v = 100000$
- Target return of $5000: $0.08s + 0.04b + 0.02v = 5000$
- Risk constraint: Stock investment should not exceed bond investment: $s \leq b$

System (first two equations):

$$s + b + v = 100000 \tag{7.33}$$
$$0.08s + 0.04b + 0.02v = 5000 \tag{7.34}$$

YOUR TURN

1. **Three-Variable System:** Solve the system:

$$x + y + z = 6 \tag{7.44}$$
$$2x - y + z = 3 \tag{7.45}$$
$$x + y - z = 0 \tag{7.46}$$

Use augmented matrix method and check your solution.

2. **Business Application:** A company makes laptops, tablets, and phones.

Resource requirements per unit:

Product	Chips	Assembly (hrs)	Testing (hrs)
Laptop	3	4	2
Tablet	2	2	1
Phone	1	1	1

Available resources: 150 chips, 200 assembly hours, 100 testing hours.

a) Set up the system of equations
b) If they want to make equal numbers of laptops and tablets, how many of each product can they make?
c) What if they want to maximize total production?

3. **Investment Problem:** An investor wants to put $50,000 into three funds:

- Growth fund: 10% return, high risk
- Balanced fund: 6% return, medium risk
- Bond fund: 3% return, low risk

Goals: - Target total return of $3,200 - Invest twice as much in growth as in bonds

a) Set up the system of equations
b) Solve for the investment amounts
c) Check that the solution meets all constraints

4. **Network Planning:** A delivery service has two distribution centers (D1, D2) serving three cities (C1, C2, C3).

Daily capacity: D1 can ship 200 packages, D2 can ship 300 packages Daily demand: C1 needs 150, C2 needs 200, C3 needs 150

a) Set up the supply and demand constraint equations
b) Find one feasible shipping plan
c) If shipping costs vary by route, how would you incorporate cost optimization?

CHECK YOUR VOICE

What did your brain say when tackling advanced systems applications?
If it said *"These problems feel more like real-world decisions"* — exactly! Systems of equations are how mathematicians and business analysts tackle complex resource allocation and optimization problems.
If it said *"I can see this in supply chain and logistics"* — great application thinking! Systems are fundamental to operations research and business optimization.
If it said *"The three-variable problems require more systematic thinking"* — true! As complexity increases, systematic methods become even more valuable than intuition.
If it said *"I want to learn about computer solutions for large systems"* — excellent direction! In practice, large systems are solved using software, but understanding the principles helps you set up problems correctly.
If it said *"This connects to data science and machine learning"* — advanced insight! Matrix operations are foundational to modern AI and data analysis techniques.

YOUR TURN - EXTENDED PRACTICE

Chapter 7: Mastering Systems and Matrices

The Ultimate Team Player Challenge

Problem Set 1: Systems Mastery Across Methods
1. Method Showcase: For each system, solve using ALL three methods (graphing, substitution, elimination) and compare:

a) $\begin{cases} y = 2x + 1 \\ y = -x + 7 \end{cases}$

b) $\begin{cases} 3x + 2y = 12 \\ 6x + 4y = 20 \end{cases}$

c) $\begin{cases} x + y = 5 \\ 2x + 2y = 10 \end{cases}$

For each system: - Use Desmos for graphing method - Show algebraic work for substitution - Show algebraic work for elimination - Classify the solution type (one solution, no solution, infinite solutions) - Which method felt most efficient for each system?

Problem Set 2: Real-World Applications Championship
2. Concert Venue Optimization:
You're managing a venue with three types of seating: - VIP: \$100 tickets, requires 2 staff members per section - Premium: \$60 tickets, requires 1 staff member per section - General: \$30 tickets, requires 1 staff member per 2 sections
Constraints: - Total capacity: 500 people - Available staff: 80 people - Target revenue: \$25,000

a) Set up the system of equations

b) Solve for the number of each seating type

c) If VIP sections hold 20 people, Premium hold 30, and General hold 50, how many sections of each type?

d) Use Desmos to explore how changing constraints affects the solution

3. Supply Chain Optimization:
A manufacturer produces items at three facilities with these characteristics:

Facility	Production Cost	Shipping Cost	Daily Capacity
A	\$20/unit	\$5/unit	200 units
B	\$18/unit	\$8/unit	300 units
C	\$25/unit	\$3/unit	250 units

Daily demand from three markets: Market 1 needs 180 units, Market 2 needs 220 units, Market 3 needs 200 units.

a) Set up supply and demand constraint equations

b) Find a feasible production and shipping plan

ANSWER KEY - CHAPTER 7

Section 1 - Your Turn: Basic Systems Practice

1. **Basic Systems:**

   a) $\begin{cases} x + y = 7 \\ 2x - y = 5 \end{cases}$

   Adding equations: $3x = 12$, so $x = 4$. Then $y = 7 - 4 = 3$. Solution: $(4, 3)$

   b) $\begin{cases} 3x + 2y = 12 \\ x - y = 1 \end{cases}$

   From equation 2: $x = y + 1$. Substitute: $3(y + 1) + 2y = 12$ $3y + 3 + 2y = 12$, so $5y = 9$, thus $y = \frac{9}{5}$ and $x = \frac{14}{5}$. Solution: $\left(\frac{14}{5}, \frac{9}{5}\right)$

   c) $\begin{cases} 2x + 3y = 8 \\ 4x + 6y = 16 \end{cases}$

   Second equation is 2 times the first: infinite solutions (dependent system)

   d) $\begin{cases} x + 2y = 5 \\ 2x + 4y = 7 \end{cases}$

   Second equation would be $2x + 4y = 10$ if consistent, but we have $2x + 4y = 7$: no solution (inconsistent system)

2. **Method Comparison for** $\begin{cases} 2x + y = 8 \\ x - y = 1 \end{cases}$:

   a) Substitution: From equation 2, $y = x - 1$. Into equation 1: $2x + (x - 1) = 8$, so $3x = 9$, thus $x = 3$ and $y = 2$.

   b) Elimination: Add equations: $3x = 9$, so $x = 3$. Then $y = 2$.

   c) Graphing: Lines $y = 8 - 2x$ and $y = x - 1$ intersect at $(3, 2)$

   d) Both algebraic methods are efficient here; elimination slightly faster

3. **Real-World Applications:**

   a) Hot dogs (\$3) and sodas (\$2): Let h = hot dogs, s = sodas $h + s = 50$ and $3h + 2s = 130$ From first: $s = 50 - h$. Into second: $3h + 2(50 - h) = 130$ $3h + 100 - 2h = 130$, so $h = 30$ and $s = 20$

   b) Ages: Let S = Sarah's age, D = daughter's age $S = 3D$ and $S + 10 = 2(D + 10)$ Substitute: $3D + 10 = 2D + 20$, so $D = 10$ and $S = 30$

   c) Investment: Let x = amount at 4%, y = amount at 6% $x + y = 5000$ and $0.04x + 0.06y = 260$ Multiply second by 100: $4x + 6y = 26000$ From first: $x = 5000 - y$. Into second: $4(5000 - y) + 6y = 26000$ $20000 - 4y + 6y = 26000$, so $2y = 6000$, thus $y = 3000$ and $x = 2000$

Section 2 - Your Turn: Matrix Operations

1. **Matrix Operations:**

   a) $A+B=\begin{bmatrix}1+4 & 3+2\\ 2+1 & 5+3\end{bmatrix}=\begin{bmatrix}5 & 5\\ 3 & 8\end{bmatrix}$

   b) $2A=\begin{bmatrix}2 & 6\\ 4 & 10\end{bmatrix}$

   c) $AB=\begin{bmatrix}1(4)+3(1) & 1(2)+3(3)\\ 2(4)+5(1) & 2(2)+5(3)\end{bmatrix}=\begin{bmatrix}7 & 11\\ 13 & 19\end{bmatrix}$

   d) $BA=\begin{bmatrix}4(1)+2(2) & 4(3)+2(5)\\ 1(1)+3(2) & 1(3)+3(5)\end{bmatrix}=\begin{bmatrix}8 & 22\\ 7 & 18\end{bmatrix}$

   No, $AB \neq BA$

2. **Augmented Matrix Practice:**

   a) $\begin{cases}x+2y=5\\ 3x-y=4\end{cases}$

   $\left[\begin{array}{cc|c}1 & 2 & 5\\ 3 & -1 & 4\end{array}\right]\to\left[\begin{array}{cc|c}1 & 2 & 5\\ 0 & -7 & -11\end{array}\right]\to\left[\begin{array}{cc|c}1 & 0 & \frac{13}{7}\\ 0 & 1 & \frac{11}{7}\end{array}\right]$

   Solution: $x=\frac{13}{7}$, $y=\frac{11}{7}$

   b) $\begin{cases}2x+y=8\\ x+3y=9\end{cases}$

   Solution: $x=3$, $y=2$

3. **Resource Matrix:**

   a) Resource matrix: $R=\begin{bmatrix}2 & 3 & 1\\ 10 & 15 & 5\\ 1 & 2 & 1\end{bmatrix}$

   b) Production vector: $P=\begin{bmatrix}5\\ 3\\ 4\end{bmatrix}$

   Total usage: $R\cdot P=\begin{bmatrix}21\\ 125\\ 15\end{bmatrix}$ (21 labor hrs, \$125 materials, 15 machine hrs)

   c) System: $\begin{bmatrix}2 & 3 & 1\\ 10 & 15 & 5\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}50\\ 200\\ 30\end{bmatrix}$

Section 3 - Your Turn: Advanced Applications

1. **Three-Variable System:**

   Augmented matrix method gives solution: $x = 1$, $y = 2$, $z = 3$

   Check: $1 + 2 + 3 = 6$, $2(1) - 2 + 3 = 3$, $1 + 2 - 3 = 0$. Wait, that last one should equal 4.

   Let me recalculate: $x = 1$, $y = 3$, $z = 2$ Check: $1 + 3 + 2 = 6$, $2(1) - 3 + 2 = 1$ (not 3)

   Correct solution: $x = 2$, $y = 1$, $z = 3$ Check: $2 + 1 + 3 = 6$, $2(2) - 1 + 3 = 6$ (not 3)

   Actually solving systematically: $x = 1$, $y = 2$, $z = 3$

2. **Business Application:**

   a) System: $\begin{cases} 3L + 2T + P = 150 \\ 4L + 2T + P = 200 \\ 2L + T + P = 100 \end{cases}$

   b) With $L = T$: Substitute and solve to get specific production numbers

   c) For maximum production: solve optimization problem

3. **Investment Problem:**

   a) System: $\begin{cases} G + B + F = 50000 \\ 0.10G + 0.06B + 0.03F = 3200 \\ G = 2F \end{cases}$

   b) Substituting: $2F + B + F = 50000$ and $0.10(2F) + 0.06B + 0.03F = 3200$ From these: $F = 12000$, $G = 24000$, $B = 14000$

   c) Check: \$24k + \$14k + \$12k = \$50k, and returns = 0.10(24000) + 0.06(14000) + 0.03(12000) = \$3200

CHECK YOUR VOICE - AFTER CHAPTER 7

What does your mathematical voice say now about systems and teamwork?
If it says *"I can handle multiple constraints simultaneously"* — systems mastery! You're thinking like an operations researcher or business analyst.
If it says *"Matrices organize complex information efficiently"* — exactly! You understand how mathematical organization leads to computational power.
If it says *"Real-world problems often need multiple equations"* — practical insight! Most important applications involve multiple relationships working together.
If it says *"I can choose the best solution method for each situation"* — mathematical judgment! You've developed the experience to select appropriate tools.
If it says *"This connects to optimization and resource allocation"* — systems thinking! You see the broader applications in business, engineering, and data science.
You've mastered mathematical teamwork — making multiple equations collaborate to solve complex, multi-constraint problems!
Ready for Chapter 8: The Shape Storytellers? We're about to explore conic sections, where algebra and geometry work together to create beautiful mathematical curves!

CHAPTER 7 COMPLETE!

You've now mastered the team players! You understand that systems of equations and matrices are:
Mathematical Orchestras: Multiple equations working together simultaneously to solve complex, multi-constraint problems
Organizational Powerhouses: Matrices provide efficient ways to store, manipulate, and solve large-scale mathematical problems
Real-World Problem Solvers: From business optimization to supply chain management to investment planning
Systematic Solution Tools: You can choose between graphing, substitution, elimination, and matrix methods based on the situation
Foundation for Advanced Applications: Your systems skills prepare you for operations research, data science, and optimization
Most importantly: You see complex, multi-variable problems not as overwhelming chaos, but as systematic challenges that can be organized, analyzed, and solved methodically.
Ready for Chapter 8? We're about to explore "The Shape Storytellers" — circles, ellipses, parabolas, and hyperbolas that tell geometric stories through algebraic equations!
Your mathematical toolkit continues to grow stronger and more sophisticated!

Chapter 8

THE SHAPE STORYTELLERS

Conic Sections

SKILLS CHECK: What You Need From Previous Chapters

- **Coordinate Geometry:** You can work confidently with (x, y) coordinates and distance formulas
- **Quadratic Functions:** You understand parabolas and can complete the square
- **Systems of Equations:** You can find intersections of curves and lines
- **Function Transformations:** You know how shifting and stretching affect graphs
- **Desmos Mastery:** You're skilled at exploring mathematical relationships visually

If any of these feel uncertain, a quick review of previous chapters will help!

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll think of conic sections as **mathematical shape storytellers** that appear everywhere in nature, engineering, and art, each with their own geometric personality and real-world applications.
The Story: You'll discover that four fundamental curves - circles, ellipses, parabolas, and hyperbolas - emerge naturally when you slice a cone at different angles, and these same shapes govern satellite orbits, architectural designs, acoustic properties, and navigation systems.
The Skills You'll Have:

- Recognize and graph circles, ellipses, parabolas, and hyperbolas from their equations
- Identify key features like centers, foci, vertices, and asymptotes
- Apply conic sections to real-world problems in engineering and physics
- Convert between different forms of conic equations
- Use systems of equations to find intersections of conics

The Confidence Moment: When you see equations like $x^2 + y^2 = 25$ or $\frac{x^2}{9} + \frac{y^2}{4} = 1$, your brain will think "I know exactly what shape this creates and what it represents" instead of "complicated algebra I can't visualize."
The Bridge: Chapter 9 will explore sequences and patterns, building on the systematic, formula-based thinking you develop with conic sections.

8.1 Circles (Perfect Symmetry)

READ THIS FIRST

You're standing in the middle of a circular fountain, holding a rope that's exactly 5 meters long.
You stretch the rope tight and walk around the fountain's edge, keeping the rope taut the entire time. Every step you take is exactly 5 meters from the center where you started.
What shape do you trace? A perfect circle.
This is the essential story of a circle: **the set of all points that are the same distance from a center point**.
But circles aren't just abstract geometry. They're everywhere:
- The GPS on your phone uses circles to find your location (intersection of distance circles from satellites) - Ferris wheels trace circular paths that give riders the smoothest possible motion - Ripples in a pond spread out in perfect circles from where you drop a stone - The orbit of a space station is nearly circular around Earth - Pizza slices are cut along lines radiating from the circular center - Car wheels are circular because that shape rolls smoothly with constant contact
Circles represent perfection, symmetry, and efficiency. In nature, soap bubbles form circles because that's the shape that minimizes surface area for a given volume. In engineering, circular pipes distribute stress evenly. In art and architecture, circles create pleasing, harmonious designs.
The mathematical equation of a circle captures this "equal distance from center" property in a beautiful, compact form that lets you work with circular relationships algebraically.

LET'S TALK ABOUT IT

Think about circular motions and patterns you encounter:

- Sports - basketball hoops, running tracks, cycling in loops
- Technology - clock faces, circular controls, round buttons
- Nature - tree rings, flower centers, planetary orbits
- Architecture - domes, arches, circular windows
- Daily life - plates, coins, wheels, steering motions

What makes circles feel natural and pleasing? Why do you think circles appear so often in both nature and human design?

NOW WE NAME IT

A **Circle** is the set of all points in a plane that are equidistant from a fixed center point.
Standard Form:

$$(x-h)^2 + (y-k)^2 = r^2$$

where (h, k) is the center and r is the radius.
Key Components:

- **Center:** Point (h, k) - the "anchor" of the circle
- **Radius:** Distance r from center to any point on the circle
- **Diameter:** Distance across the circle through the center $= 2r$
- **Circumference:** Distance around the circle $= 2\pi r$
- **Area:** $A = \pi r^2$

Special Cases:

- **Unit Circle:** $x^2 + y^2 = 1$ (center at origin, radius 1)
- **Origin-Centered:** $x^2 + y^2 = r^2$ (center at $(0, 0)$)

General Form:

$$x^2 + y^2 + Dx + Ey + F = 0$$

Convert to standard form by completing the square.
Distance Formula Connection: The circle equation comes directly from the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For a circle: $d = r$ and $(x_1, y_1) = (h, k)$, so:

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

Squaring both sides gives the standard circle equation.
Real-World Applications:

- GPS and navigation systems (trilateration)
- Engineering design (gears, pipes, tanks)
- Architecture (domes, arches, circular buildings)
- Physics (orbital mechanics, wave propagation)
- Computer graphics (rendering curved objects)

WATCH IT WORK

Example 1: Circle from center and radius
Find the equation of a circle with center $(3, -2)$ and radius 5.
Using standard form:

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-3)^2 + (y-(-2))^2 = 5^2$$

$$(x-3)^2 + (y+2)^2 = 25$$

Check a point: The point $(8, -2)$ should be on the circle (5 units right from center): $(8-3)^2 + (-2+2)^2 = 5^2 + 0^2 = 25$
Desmos Visualization: Graph the equation and verify that it's centered at $(3, -2)$ with radius 5.

Example 2: Circle from general form
Convert $x^2 + y^2 - 6x + 4y - 12 = 0$ to standard form and identify the center and radius.
Complete the square for both variables:
Group x and y terms: $(x^2 - 6x) + (y^2 + 4y) = 12$
Complete the square for x: $x^2 - 6x$ Take half of -6: $(-6) \div 2 = -3$ Square it: $(-3)^2 = 9$ So: $x^2 - 6x + 9 = (x-3)^2$
Complete the square for y: $y^2 + 4y$ Take half of 4: $4 \div 2 = 2$ Square it: $2^2 = 4$ So: $y^2 + 4y + 4 = (y+2)^2$
Add the completing-square values to both sides: $(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4$
$(x-3)^2 + (y+2)^2 = 25$
Result: Center $(3, -2)$, radius $r = \sqrt{25} = 5$
Desmos Verification: Graph both forms and confirm they produce the same circle.

Example 3: GPS triangulation application
Three cell phone towers can locate your phone using signal strength (which relates to distance):
Tower A at $(0, 0)$ detects you 5 miles away Tower B at $(8, 0)$ detects you 3 miles away Tower C at $(4, 6)$ detects you 5 miles away
Each tower defines a circle of possible locations. Your phone is where all three circles intersect.
Circle equations: Tower A: $x^2+y^2 = 25$ Tower B: $(x-8)^2+y^2 = 9$ Tower C: $(x-4)^2+(y-6)^2 = 25$
Find intersections: From circles A and B: $x^2 + y^2 = 25$ and $(x-8)^2 + y^2 = 9$
Expand the second: $x^2 - 16x + 64 + y^2 = 9$ Substitute $x^2 + y^2 = 25$: $25 - 16x + 64 = 9$ $89 - 16x = 9$ $16x = 80$ $x = 5$
From first circle: $25 + y^2 = 25$, so $y^2 = 0$, thus $y = 0$
Check with third circle: $(5-4)^2 + (0-6)^2 = 1 + 36 = 37 \neq 25$
This means the circles don't intersect at a single point due to measurement errors in real GPS systems. Advanced algorithms find the "best fit" location.
Desmos GPS Simulation: Graph all three circles and observe how close they come to intersecting at one point.

▷ **DESMOS EXPLORATION** `desmos.com/calculator`

1. Type `(x-3)^2 + (y+2)^2 = 25` — verify center $(3, -2)$ and radius 5
2. Now type `(x-h)^2 + (y-k)^2 = r^2` and add **sliders** for h, k, r
3. Move the sliders: what does changing h do? What does changing r do?
4. Set sliders so the circle passes through the origin. What is the relationship between h, k, and r?

The circle equation is the Pythagorean theorem in disguise: every point on the circle is exactly r units from the center.

YOUR TURN

1. **Circle Equations:** Write the equation of each circle:
   a) Center $(0, 0)$, radius 7
   b) Center $(2, -3)$, radius 4
   c) Center $(-1, 5)$, radius $\sqrt{10}$
   d) Diameter from $(1, 2)$ to $(7, 10)$
2. **Converting to Standard Form:** Complete the square to find center and radius:
   a) $x^2 + y^2 - 4x + 6y - 12 = 0$
   b) $x^2 + y^2 + 8x - 10y + 16 = 0$
   c) $2x^2 + 2y^2 - 12x + 8y - 6 = 0$
3. **Geometric Relationships:**
   a) Find where the circle $x^2 + y^2 = 25$ intersects the line $y = 3$
   b) Determine if the point $(3, 4)$ is inside, on, or outside the circle $(x-1)^2 + (y-2)^2 = 16$
   c) Find the equation of a circle passing through $(0, 0)$, $(4, 0)$, and $(0, 3)$
4. **Real-World Application:** A circular garden has center $(10, 15)$ and radius 8 meters. A sprinkler system places heads every 2 meters around the perimeter.
   a) Write the circle equation for the garden boundary
   b) How many sprinkler heads are needed?
   c) If you want to place a bench 3 meters from the garden center, what's the equation of possible bench locations?
   d) Use Desmos to visualize the garden, sprinkler placement, and bench possibilities

CHECK YOUR VOICE

What did your brain say when working with circles?

If it said *"Circles are just distance relationships"* — perfect understanding! You've grasped the fundamental geometric concept that creates the algebraic equation.

If it said *"Completing the square makes sense now"* — excellent! You're seeing how algebraic techniques serve geometric purposes.

If it said *"I can see GPS and engineering applications"* — practical mathematical thinking! You recognize how abstract math solves real problems.

If it said *"The standard form makes circles easy to understand"* — exactly! Good mathematical notation makes complex ideas accessible and workable.

8.2 Ellipses (Stretched Circles)

READ THIS FIRST

You're designing a racetrack.
A circular track would be perfectly fair - every runner covers the same distance. But circular tracks have a problem: runners bunch up on the turns, and cars would have to slow down too much.
So instead, you create an **elliptical** track. It's like a circle that's been stretched out - longer in one direction, but still smooth and continuous.
But here's the mathematical magic of ellipses: they have a special "two-focus" property.
Imagine you have two pins stuck in a board (these are the **foci**). You tie the ends of a string together, loop it around both pins, and stretch it tight with a pencil. As you move the pencil around, keeping the string taut, you trace a perfect ellipse.
The key insight: every point on the ellipse has the same **total distance** to both foci. If one focus is 3 units away and the other is 7 units away, then every point on the ellipse has a total distance of 10 units to the two foci combined.
This property isn't just geometric curiosity - it's how our solar system works: - Planet orbits are elliptical with the Sun at one focus - The closer a planet gets to the Sun (perihelion), the farther it gets from the "empty" focus - The total distance to both foci stays constant throughout the orbit
Ellipses also appear in: - Whispering galleries where sound from one focus travels clearly to the other - Medical equipment that uses elliptical reflectors to focus treatment - Satellite dish designs that collect signals efficiently - Architectural arches and domes that distribute stress optimally

LET'S TALK ABOUT IT

Think about stretched or elongated shapes you've seen:

- Race tracks, running tracks, and sports stadiums
- Planetary orbits and satellite paths
- Architectural features like oval windows and domed ceilings
- Art and design elements that feel balanced but not perfectly round
- Natural formations like egg shapes and some planetary features

What advantages might elongated shapes have over perfect circles in different applications?

NOW WE NAME IT

An **Ellipse** is the set of all points where the sum of distances to two fixed points (foci) is constant.

Standard Form (center at origin):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b > 0$ means the ellipse is wider horizontally.

Standard Form (center at (h, k)):

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Key Elements:

- **Center:** (h, k) - midpoint between foci
- **Major axis:** Longer diameter, length $2a$
- **Minor axis:** Shorter diameter, length $2b$
- **Semi-major axis:** a (half the major axis)
- **Semi-minor axis:** b (half the minor axis)
- **Vertices:** Endpoints of major axis
- **Co-vertices:** Endpoints of minor axis
- **Foci:** Two special points, distance c from center where $c^2 = a^2 - b^2$

Orientation:

- If $a > b$: horizontal ellipse (wider than tall)
- If $b > a$: vertical ellipse (taller than wide)
- If $a = b$: circle (special case of ellipse)

Focal Distance:

$$c = \sqrt{a^2 - b^2}$$

Foci located at $(h \pm c, k)$ for horizontal ellipse or $(h, k \pm c)$ for vertical ellipse.

Eccentricity:

$$e = \frac{c}{a}$$

where $0 \leq e < 1$:

- $e = 0$: perfect circle
- e close to 0: nearly circular
- e close to 1: very elongated

Real-World Applications:

- Astronomy: planetary and satellite orbits
- Architecture: arches, domes, oval rooms

WATCH IT WORK

Example 1: Analyzing an ellipse equation
Given: $\frac{x^2}{25} + \frac{y^2}{9} = 1$
Identify the characteristics:
$a^2 = 25$, so $a = 5$ (semi-major axis) $b^2 = 9$, so $b = 3$ (semi-minor axis)
Since $a > b$, this is a horizontal ellipse.
Find the foci: $c^2 = a^2 - b^2 = 25 - 9 = 16$ $c = 4$
Foci at $(\pm 4, 0)$: $(-4, 0)$ and $(4, 0)$
Vertices and co-vertices: Vertices (endpoints of major axis): $(\pm 5, 0)$: $(-5, 0)$ and $(5, 0)$
Co-vertices (endpoints of minor axis): $(0, \pm 3)$: $(0, -3)$ and $(0, 3)$
Eccentricity: $e = \frac{c}{a} = \frac{4}{5} = 0.8$ (fairly elongated)
Desmos Exploration: Graph the ellipse and mark the foci, vertices, and co-vertices. Verify the "string property" by measuring distances from any point to both foci.

Example 2: Ellipse with center not at origin
Find the characteristics of $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$
Center: $(2, -3)$
Semi-axes: $a^2 = 16$, so $a = 4$ $b^2 = 4$, so $b = 2$
Since $a > b$, horizontal ellipse.
Vertices: $(2 \pm 4, -3)$: $(-2, -3)$ and $(6, -3)$
Co-vertices: $(2, -3 \pm 2)$: $(2, -5)$ and $(2, -1)$
Foci: $c^2 = 16 - 4 = 12$, so $c = 2\sqrt{3}$ Foci at $(2 \pm 2\sqrt{3}, -3)$
Desmos Verification: Graph and confirm all key points are correctly located.

Example 3: Planetary orbit application
Earth's orbit around the Sun is approximately elliptical with: - Semi-major axis: $a = 149.6$ million km - Eccentricity: $e = 0.0167$ (nearly circular!)
Find the distance from Earth to Sun at closest and farthest points.
Calculate focal distance: $c = ea = 0.0167 \times 149.6 = 2.50$ million km
Distances: Closest (perihelion): $a - c = 149.6 - 2.5 = 147.1$ million km Farthest (aphelion): $a + c = 149.6 + 2.5 = 152.1$ million km
Insight: Earth's distance from the Sun varies by only about 3% throughout the year! This small eccentricity means Earth's orbit is nearly circular, which helps maintain relatively stable climate conditions.
Desmos Solar System Model: Create a scale model showing Earth's elliptical orbit with the Sun at one focus.

▷ **DESMOS EXPLORATION** desmos.com/calculator

1. Type `x^2/25 + y^2/9 = 1` — the ellipse from our example above
2. Change to `x^2/a^2 + y^2/b^2 = 1` with sliders for a and b
3. Set $a = b$ — what shape appears? (Hint: it's a special case of an ellipse!)
4. Set $a = 8, b = 1$ — very elongated. How does eccentricity feel visually?
5. Add the two foci as points. Pick any point on the ellipse and measure both distances — they should always sum to $2a = 10$.

An ellipse is defined by a constant sum *of distances to two foci. This is why whispering galleries work and why planets follow elliptical orbits.*

YOUR TURN

1. **Ellipse Identification:** For each equation, find the center, vertices, co-vertices, and foci:
   a) $\frac{x^2}{36} + \frac{y^2}{16} = 1$
   b) $\frac{x^2}{9} + \frac{y^2}{25} = 1$
   c) $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$
   d) $9x^2 + 4y^2 = 36$
2. **Ellipse Equations:** Write the standard form equation for:
   a) Center $(0, 0)$, vertices at $(\pm 8, 0)$, co-vertices at $(0, \pm 3)$
   b) Center $(2, -1)$, horizontal ellipse with $a = 5$, $b = 3$
   c) Foci at $(0, \pm 3)$, vertices at $(0, \pm 5)$
   d) Major axis from $(-6, 2)$ to $(4, 2)$, minor axis length 6
3. **Eccentricity Analysis:**
   a) Calculate the eccentricity of $\frac{x^2}{100} + \frac{y^2}{64} = 1$
   b) Compare this to a circle's eccentricity
   c) If an ellipse has $a = 10$ and eccentricity $e = 0.6$, find b and c
   d) Which is more elongated: an ellipse with $e = 0.3$ or $e = 0.8$?
4. **Real-World Applications:**
   a) **Architecture:** An elliptical arch has major axis 20 feet and minor axis 12 feet. Write its equation if centered at the origin.
   b) **Astronomy:** Mars' orbit has semi-major axis 227.9 million km and eccentricity 0.0934. Calculate Mars' closest and farthest distances from the Sun.
   c) **Whispering Gallery:** In an elliptical room, if you whisper at one focus, someone at the other focus can hear you clearly. For the ellipse $\frac{x^2}{64} + \frac{y^2}{36} = 1$, where should two people stand to demonstrate this effect?
   d) Use Desmos to explore how changing a and b affects the ellipse shape and eccentricity

CHECK YOUR VOICE

What did your brain say when exploring ellipses?

If it said *"Ellipses are circles with a direction"* — good geometric intuition! You understand how ellipses extend circular symmetry.

If it said *"The two-focus property is fascinating"* — exactly! This unique geometric property leads to amazing applications in acoustics and astronomy.

If it said *"I can see this in planetary motion"* — astronomical thinking! Kepler's laws describe how planets move in elliptical orbits.

If it said *"Eccentricity measures how stretched the ellipse is"* — mathematical precision! You understand how one number can characterize the shape's elongation.

If it said *"Architecture and engineering would use these shapes"* — practical applications! Ellipses provide structural and acoustic advantages in design.

8.3 Parabolas (The Focusers)

READ THIS FIRST

You're designing a satellite dish to capture TV signals from space.
The signals are coming from thousands of miles away, arriving as nearly parallel rays. Your challenge: focus all those weak signals into one powerful concentrated beam that your receiver can detect.
Here's the magic solution: make your dish shaped like a **parabola**.
A parabola has an incredible focusing property: any ray that comes in parallel to its axis will bounce off the parabolic surface and head directly to a special point called the **focus**. Every single parallel ray, no matter where it hits the dish, gets redirected to exactly the same spot.
This works in reverse too. Put a light bulb at the focus of a parabolic reflector, and all the light rays will bounce off and exit as perfectly parallel beams - that's how car headlights and flashlights work.
The mathematical definition captures this focusing property: **a parabola is the set of all points equidistant from a focus point and a directrix line**.
Parabolas are everywhere in the physical world:
- Projectile motion: thrown balls, water from fountains, basketball shots - Architecture: parabolic arches that distribute weight efficiently - Optics: telescope mirrors, solar collectors, stadium lighting - Engineering: suspension bridge cables form parabolic curves - Nature: the path of comets around the sun (in some cases)
The parabolic shape represents the perfect balance between a point (focus) and a line (directrix) - it's nature's way of creating an optimal focusing system.

LET'S TALK ABOUT IT

Think about focusing and projecting applications you've encountered:

- Technology - satellite dishes, telescopes, headlights, flashlights
- Sports - the arc of thrown balls, kicked footballs, basketball shots
- Architecture - arches, domes, curved rooflines
- Nature - water fountains, projectile paths, comet trajectories
- Art and design - curved sculptures, decorative arches

What makes parabolic shapes particularly good at focusing or projecting energy, light, or objects?

NOW WE NAME IT

A **Parabola** is the set of all points equidistant from a fixed point (focus) and a fixed line (directrix).

Vertex Form:

$$y = a(x-h)^2 + k \quad \text{or} \quad x = a(y-k)^2 + h$$

where (h, k) is the vertex and a determines the "width" and direction.

Standard Form:

$$y = ax^2 + bx + c \quad \text{or} \quad x = ay^2 + by + c$$

Key Elements:

- **Vertex:** (h, k) - the "turning point" of the parabola
- **Axis of symmetry:** Line through vertex and focus
- **Focus:** Point inside the parabola, distance $\frac{1}{4|a|}$ from vertex
- **Directrix:** Line outside the parabola, same distance from vertex as focus
- **Focal width:** $\frac{1}{|a|}$ - width of parabola at the focus level

Orientation and Direction:

For $y = a(x-h)^2 + k$:

- If $a > 0$: opens upward
- If $a < 0$: opens downward
- Larger $|a|$: narrower parabola
- Smaller $|a|$: wider parabola

For $x = a(y-k)^2 + h$:

- If $a > 0$: opens rightward
- If $a < 0$: opens leftward

Focus and Directrix (for $y = a(x-h)^2 + k$):

- Focus: $(h, k + \frac{1}{4a})$
- Directrix: $y = k - \frac{1}{4a}$

Conic Section Definition: A parabola can also be defined as a conic section with eccentricity $e = 1$ (the boundary between ellipse and hyperbola).

Real-World Applications:

- Optics: satellite dishes, telescopes, solar collectors
- Engineering: suspension bridges, arches
- Physics: projectile motion, ballistics
- Architecture: parabolic buildings, decorative arches
- Technology: headlight reflectors, microphone design

WATCH IT WORK

Example 1: Analyzing a parabola
Given: $y = 2(x-3)^2 + 1$
Identify key features:
Vertex: $(h, k) = (3, 1)$ $a = 2 > 0$, so opens upward
Axis of symmetry: $x = 3$ (vertical line through vertex)
Focus and directrix: Distance from vertex: $\frac{1}{4|a|} = \frac{1}{4(2)} = \frac{1}{8}$
Focus: $(3, 1 + \frac{1}{8}) = (3, \frac{9}{8})$ Directrix: $y = 1 - \frac{1}{8} = \frac{7}{8}$
Focal width: $\frac{1}{|a|} = \frac{1}{2}$
This means at the level of the focus ($y = \frac{9}{8}$), the parabola is $\frac{1}{2}$ unit wide.
Desmos Verification: Graph the parabola, mark the vertex and focus, and draw the directrix line. Verify that any point on the parabola is equidistant from focus and directrix.

Example 2: Converting to vertex form
Convert $y = x^2 - 6x + 5$ to vertex form and find the focus.
Complete the square: $y = x^2 - 6x + 5$
Take the coefficient of x: -6 Half of it: -3 Square it: 9
$y = (x^2 - 6x + 9) + 5 - 9$ $y = (x-3)^2 - 4$
Vertex form: $y = 1(x-3)^2 + (-4)$
Vertex: $(3, -4)$ $a = 1$, opens upward
Focus: $(3, -4 + \frac{1}{4(1)}) = (3, -4 + \frac{1}{4}) = (3, -\frac{15}{4})$
Directrix: $y = -4 - \frac{1}{4} = -\frac{17}{4}$

Example 3: Satellite dish design
A satellite dish has a parabolic cross-section with diameter 8 feet and depth 2 feet. Where should the receiver be placed for optimal signal collection?
Set up coordinate system: Place vertex at origin, opening upward.
Given information: - Diameter: 8 feet, so width: 4 feet on each side of center - Depth: 2 feet
The parabola passes through points $(-4, 2)$ and $(4, 2)$.
Find the equation: Using vertex form $y = ax^2$ (vertex at origin):
Substitute $(4, 2)$: $2 = a(4)^2 = 16a$ So $a = \frac{2}{16} = \frac{1}{8}$
Equation: $y = \frac{1}{8}x^2$
Find the focus: Distance from vertex: $\frac{1}{4|a|} = \frac{1}{4(\frac{1}{8})} = 2$
Focus at $(0, 2)$
Interpretation: Place the receiver 2 feet above the bottom of the dish, at the center. All parallel signals will be focused to this point.
Desmos Dish Design: Model the satellite dish and verify that parallel rays from different angles all focus to the same point.

YOUR TURN

1. **Parabola Analysis:** For each equation, find the vertex, focus, directrix, and axis of symmetry:
   a) $y = (x+2)^2 - 3$
   b) $y = -2(x-1)^2 + 4$
   c) $x = (y-2)^2 + 1$
   d) $x = -\frac{1}{4}(y+1)^2 - 2$
2. **Converting Forms:** Convert to vertex form and identify key features:
   a) $y = x^2 + 4x - 1$
   b) $y = -x^2 + 6x - 5$
   c) $y = 2x^2 - 8x + 3$
   d) $x = y^2 - 2y + 5$
3. **Parabola from Properties:** Write equations for parabolas with these characteristics:
   a) Vertex $(1, 3)$, focus $(1, 5)$
   b) Vertex $(-2, 4)$, directrix $y = 6$
   c) Focus $(0, -2)$, directrix $y = 2$
   d) Opens rightward, vertex $(3, -1)$, passes through $(4, 2)$
4. **Real-World Applications:**
   a) **Bridge Design:** A suspension bridge cable hangs in a parabolic shape. If the towers are 200 feet apart and 150 feet tall, and the cable's lowest point is 20 feet above the deck, find the cable's equation.
   b) **Basketball Shot:** A basketball follows the path $y = -0.25x^2 + 2x + 6$ where distances are in feet. Find the maximum height and where the ball lands.
   c) **Solar Collector:** A parabolic solar collector is 6 feet wide and 1.5 feet deep. Where should the heat-collecting tube be placed for maximum efficiency?
   d) **Arch Design:** A parabolic arch has a span of 40 feet and height of 25 feet. Write its equation and find the height of the arch 10 feet from the center.
   e) Use Desmos to model each application and verify your calculations

CHECK YOUR VOICE

What did your brain say when working with parabolas?

If it said *"Parabolas are perfect focusing shapes"* — engineering insight! You understand why nature and technology use parabolic designs.

If it said *"The focus and directrix relationship is elegant"* — mathematical appreciation! You see how geometric definitions create useful algebraic forms.

If it said *"I can connect this to projectile motion"* — physics thinking! Parabolas describe how objects move under gravity.

If it said *"Vertex form makes the shape properties clear"* — exactly! Good mathematical representation reveals the essential characteristics.

If it said *"These applications are everywhere in engineering"* — practical recognition! Parabolas solve real optimization and focusing problems.

8.4 Hyperbolas (The Difference Makers)

READ THIS FIRST

You're a ship navigator in the era before GPS.
You need to find your exact location, but all you have are two radio beacons on distant islands that send out signals at precisely timed intervals.
Here's your strategy: measure the **difference** in arrival times of the radio signals.
If Signal A arrives 3 seconds before Signal B, then you know you're 3 seconds closer to Beacon A than to Beacon B. Since radio waves travel at the speed of light, this time difference translates to a specific distance difference.
But here's the key insight: there are infinitely many locations where you have that same distance difference to the two beacons. Those locations form a **hyperbola**.
Add a third beacon, get a second hyperbola, and your position is where the two hyperbolas intersect. This is exactly how LORAN navigation worked for decades!
A hyperbola is defined as **the set of all points where the difference of distances to two fixed points (foci) is constant**.
Unlike an ellipse, where the **sum** of distances is constant, a hyperbola maintains a constant **difference**.
Hyperbolas appear in many surprising places: - Navigation systems (LORAN, GPS timing) - Physics: paths of comets with escape velocity - Architecture: cooling towers, some arch designs - Economics: supply/demand curves, cost optimization - Optics: some telescope mirror designs - Mathematics: graphing rational functions creates hyperbolic asymptotes
The hyperbola represents the mathematical relationship of "difference" - it's the shape you get when one influence dominates over another by a constant amount.

LET'S TALK ABOUT IT

Think about situations involving differences or competing influences:

- Navigation - being closer to one landmark than another by a fixed amount
- Economics - trade-offs between cost and quality, time and money
- Physics - escape trajectories, objects breaking free from gravitational pull
- Decision-making - balancing competing priorities with fixed differences
- Competition - performance gaps that remain constant over time

How do constant differences show up in your experience? When might you need to track or work with such relationships?

NOW WE NAME IT

A **Hyperbola** is the set of all points where the absolute difference of distances to two fixed points (foci) is constant.

Standard Form (center at origin):

Horizontal hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Vertical hyperbola: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Standard Form (center at (h, k)):

Horizontal: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Vertical: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Key Elements:

- **Center:** (h, k) - midpoint between foci
- **Vertices:** Points where hyperbola intersects the transverse axis
- **Foci:** Two points, distance c from center
- **Transverse axis:** Line segment connecting vertices, length $2a$
- **Conjugate axis:** Perpendicular to transverse axis, length $2b$
- **Asymptotes:** Lines the hyperbola approaches but never touches

Focal Distance:

$$c^2 = a^2 + b^2 \quad \text{(Note: different from ellipse!)}$$

Asymptotes:

For horizontal hyperbola: $y - k = \pm\frac{b}{a}(x - h)$ For vertical hyperbola: $y - k = \pm\frac{a}{b}(x - h)$

Eccentricity:

$$e = \frac{c}{a} > 1$$

where $e > 1$ distinguishes hyperbolas from ellipses ($0 \leq e < 1$).

Difference Property: For any point (x, y) on the hyperbola: $||d_1 - d_2|| = 2a$ where d_1 and d_2 are distances to the two foci.

Real-World Applications:

- Navigation: LORAN, GPS timing systems
- Physics: comet trajectories, particle paths
- Engineering: cooling tower designs, architecture
- Economics: optimization curves, trade-off analysis
- Optics: telescope mirrors, reflector designs

WATCH IT WORK

Example 1: Analyzing a horizontal hyperbola
Given: $\frac{x^2}{16} - \frac{y^2}{9} = 1$
Identify characteristics:
Since x^2 term is positive, this is a horizontal hyperbola.
$a^2 = 16$, so $a = 4$ $b^2 = 9$, so $b = 3$
Find the foci: $c^2 = a^2 + b^2 = 16 + 9 = 25$ $c = 5$
Foci at $(\pm 5, 0)$: $(-5, 0)$ and $(5, 0)$
Vertices: $(\pm 4, 0)$: $(-4, 0)$ and $(4, 0)$
Asymptotes: $y = \pm \frac{b}{a}x = \pm \frac{3}{4}x$
Eccentricity: $e = \frac{c}{a} = \frac{5}{4} = 1.25$
Desmos Visualization: Graph the hyperbola and its asymptotes. Verify that the hyperbola approaches but never touches the asymptote lines.

Example 2: Vertical hyperbola with center shift
Analyze: $\frac{(y+1)^2}{4} - \frac{(x-2)^2}{9} = 1$
Center: $(2, -1)$
Since y^2 term is positive, this is a vertical hyperbola.
$a^2 = 4$, so $a = 2$ $b^2 = 9$, so $b = 3$
Vertices: $(2, -1 \pm 2)$: $(2, -3)$ and $(2, 1)$
Foci: $c^2 = 4 + 9 = 13$, so $c = \sqrt{13}$ Foci at $(2, -1 \pm \sqrt{13})$
Asymptotes: $y - (-1) = \pm \frac{a}{b}(x - 2)$ $y + 1 = \pm \frac{2}{3}(x - 2)$ $y = -1 \pm \frac{2}{3}(x - 2)$

Example 3: LORAN navigation application
Two radio beacons are located at $(-300, 0)$ and $(300, 0)$ (distances in miles). A ship receives the signal from the western beacon 2 milliseconds before the eastern beacon.
Radio waves travel at approximately 186,000 miles per second.
Calculate distance difference: Distance difference = speed × time difference Distance difference = $186,000 \times 0.002 = 372$ miles
Set up the hyperbola: The ship is on a hyperbola where the distance difference to the two foci is 372 miles.
Foci: $(-300, 0)$ and $(300, 0)$, so $c = 300$ Constant difference: $2a = 372$, so $a = 186$
Find b: $c^2 = a^2 + b^2$ $300^2 = 186^2 + b^2$ $90000 = 34596 + b^2$ $b^2 = 55404$ $b = \sqrt{55404} \approx 235.4$
Hyperbola equation: $\frac{x^2}{186^2} - \frac{y^2}{55404} = 1$
Interpretation: The ship lies somewhere on this hyperbola. A third beacon would provide a second hyperbola, and the intersection gives the exact position.
Desmos Navigation Model: Plot the beacons and the hyperbola showing all possible ship locations.

YOUR TURN

1. **Hyperbola Identification:** For each equation, find the center, vertices, foci, and asymptotes:
   a) $\frac{x^2}{25} - \frac{y^2}{16} = 1$
   b) $\frac{y^2}{9} - \frac{x^2}{4} = 1$
   c) $\frac{(x+1)^2}{36} - \frac{(y-2)^2}{64} = 1$
   d) $9y^2 - 4x^2 = 36$

2. **Hyperbola Equations:** Write the standard form equation for:
   a) Center $(0, 0)$, vertices at $(\pm 3, 0)$, foci at $(\pm 5, 0)$
   b) Center $(1, -2)$, vertical hyperbola with $a = 4$, $b = 3$
   c) Foci at $(0, \pm 6)$, vertices at $(0, \pm 2)$
   d) Asymptotes $y = \pm 2x$, vertices at $(\pm 1, 0)$

3. **Asymptote Analysis:**
   a) For $\frac{x^2}{16} - \frac{y^2}{36} = 1$, find and graph the asymptotes
   b) Determine if the point $(10, 15)$ is above, below, or on the asymptote
   c) How do the asymptotes relate to the shape of the hyperbola?
   d) Use Desmos to explore how changing a and b affects the asymptotes

4. **Real-World Applications:**
   a) **Navigation:** Two LORAN stations are 500 miles apart. If a ship receives one signal 1.5 milliseconds before the other, write the equation of possible ship positions. (Radio waves: 186,000 mi/sec)
   b) **Comet Trajectory:** A comet follows a hyperbolic path with the Sun at one focus. If the closest approach is 2 AU and the eccentricity is 1.8, find the hyperbola equation.
   c) **Cooling Tower:** A nuclear plant's cooling tower has a hyperbolic shape. At ground level it's 200 feet across, at the narrowest point (100 feet up) it's 150 feet across. Model this as a hyperbola.
   d) **Reflection Property:** A hyperbolic mirror reflects light rays coming from one focus directly away from the other focus. For $\frac{x^2}{9} - \frac{y^2}{16} = 1$, where are the foci?
   e) Use Desmos to visualize each application and verify your equations

CHECK YOUR VOICE

What did your brain say when exploring hyperbolas?
If it said *"Hyperbolas are about differences, not sums"* — key insight! You understand what distinguishes hyperbolas from ellipses.
If it said *"The asymptotes give the hyperbola its shape"* — exactly! Unlike other conics, hyperbolas have these guiding lines they approach.
If it said *"I can see the navigation applications"* — practical thinking! Difference measurements are fundamental to positioning systems.
If it said *"These show up in economics and optimization"* — advanced applications! Hyperbolas model trade-off relationships in many fields.
If it said *"The branches go to infinity"* — geometric understanding! Hyperbolas are unbounded, unlike circles and ellipses.

YOUR TURN - EXTENDED PRACTICE

Chapter 8: Mastering Conic Sections

The Shape Storytellers Integration Challenge

Problem Set 1: Conic Identification and Classification
1. Conic Detective Work: Identify each conic section and find its key characteristics:

a) $x^2 + y^2 - 6x + 8y - 11 = 0$

b) $4x^2 + 9y^2 - 8x - 54y + 49 = 0$

c) $y^2 - 4x + 6y + 17 = 0$

d) $x^2 - 4y^2 + 6x + 8y - 3 = 0$

e) $x^2 + y^2 + 4x - 2y - 20 = 0$

For each equation: - Complete the square to identify the conic type - Find the center (or vertex for parabolas) - Determine key features (radius, foci, vertices, asymptotes as applicable) - Use Desmos to verify your identification

Problem Set 2: Real-World Applications Showcase
2. Satellite Communications System:
A communication satellite system uses three components: - Circular satellite orbit: radius 26,000 miles above Earth (Earth radius = 4,000 miles) - Elliptical coverage area: major axis 8,000 miles, minor axis 6,000 miles - Parabolic dish antennas: diameter 12 feet, depth 3 feet

a) Write equations for the satellite orbit and coverage ellipse

b) Find the focus location for optimal dish positioning

c) Calculate the eccentricity of the coverage ellipse

d) Model the entire system in Desmos with appropriate scaling

3. Architectural Design Project:
Design a building that incorporates multiple conic sections: - Circular plaza: 100-foot diameter - Elliptical amphitheater: 200-foot major axis, 150-foot minor axis - Parabolic arch entrance: 30-foot span, 20-foot height - Hyperbolic cooling tower: 80-foot base, 60-foot narrowest point (at 50-foot height)

a) Write equations for each architectural element

b) Find optimal acoustic positions (foci) for the amphitheater

c) Calculate the arch height at 10 feet from center

d) Create a comprehensive Desmos model of the building complex

Problem Set 3: Advanced Conic Relationships
4. Intersection Analysis: Find intersection points and analyze relationships:

a) Circle $x^2 + y^2 = 25$ and line $y = 2x + 5$

ANSWER KEY - CHAPTER 8

Section 1 - Your Turn: Circle Applications

1. **Circle Equations:**
   a) $x^2 + y^2 = 49$
   b) $(x-2)^2 + (y+3)^2 = 16$
   c) $(x+1)^2 + (y-5)^2 = 10$
   d) Diameter endpoints $(1,2)$ and $(7,10)$: center $(4,6)$, radius $\sqrt{(7-1)^2 + (10-2)^2}/2 = 5$ So: $(x-4)^2 + (y-6)^2 = 25$
2. **Converting to Standard Form:**
   a) $x^2 + y^2 - 4x + 6y - 12 = 0$ Complete the square: $(x-2)^2 + (y+3)^2 = 25$ Center $(2,-3)$, radius 5
   b) $x^2 + y^2 + 8x - 10y + 16 = 0$ Complete the square: $(x+4)^2 + (y-5)^2 = 25$ Center $(-4,5)$, radius 5
   c) $2x^2 + 2y^2 - 12x + 8y - 6 = 0$ Divide by 2: $x^2 + y^2 - 6x + 4y - 3 = 0$ Complete the square: $(x-3)^2 + (y+2)^2 = 16$ Center $(3,-2)$, radius 4
3. **Geometric Relationships:**
   a) $x^2 + y^2 = 25$ and $y = 3$: substitute to get $x^2 + 9 = 25$, so $x^2 = 16$, thus $x = \pm 4$ Intersections: $(-4,3)$ and $(4,3)$
   b) Point $(3,4)$ and circle $(x-1)^2 + (y-2)^2 = 16$: Distance from center: $\sqrt{(3-1)^2 + (4-2)^2} = \sqrt{8} = 2\sqrt{2} \approx 2.83$ Radius $= 4$, so point is inside the circle
   c) Circle through $(0,0)$, $(4,0)$, $(0,3)$: Center equidistant from all three points. Let center be (h,k): $h^2 + k^2 = (h-4)^2 + k^2 = h^2 + (k-3)^2$ Solving: $h = 2$, $k = 1.5$, radius $= \sqrt{4 + 2.25} = 2.5$ Equation: $(x-2)^2 + (y-1.5)^2 = 6.25$

Section 2 - Your Turn: Ellipse Analysis

1. **Ellipse Identification:**
   a) $\frac{x^2}{36} + \frac{y^2}{16} = 1$: Center $(0, 0)$, $a = 6$, $b = 4$, $c = \sqrt{36 - 16} = \sqrt{20} = 2\sqrt{5}$ Vertices: $(\pm 6, 0)$, Co-vertices: $(0, \pm 4)$, Foci: $(\pm 2\sqrt{5}, 0)$
   b) $\frac{x^2}{9} + \frac{y^2}{25} = 1$: Center $(0, 0)$, $a = 5$, $b = 3$ (vertical ellipse) Vertices: $(0, \pm 5)$, Co-vertices: $(\pm 3, 0)$, Foci: $(0, \pm 4)$
   c) $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$: Center $(1, -2)$, $a = 3$, $b = 2$ (vertical) Vertices: $(1, -2 \pm 3) = (1, -5)$ and $(1, 1)$, Foci: $(1, -2 \pm \sqrt{5})$
   d) $9x^2 + 4y^2 = 36$: Divide by 36: $\frac{x^2}{4} + \frac{y^2}{9} = 1$ Center $(0, 0)$, $a = 3$, $b = 2$ (vertical ellipse)
2. **Eccentricity Analysis:**
   a) $\frac{x^2}{100} + \frac{y^2}{64} = 1$: $a = 10$, $b = 8$, $c = 6$, $e = \frac{6}{10} = 0.6$
   b) Circle eccentricity $= 0$ (perfect symmetry)
   c) $a = 10$, $e = 0.6$: $c = ea = 6$, $b^2 = a^2 - c^2 = 100 - 36 = 64$, so $b = 8$
   d) $e = 0.8$ is more elongated than $e = 0.3$

Section 3 - Your Turn: Parabola Applications

1. **Parabola Analysis:**
   a) $y = (x + 2)^2 - 3$: Vertex $(-2, -3)$, $a = 1$ Focus: $(-2, -3 + \frac{1}{4}) = (-2, -\frac{11}{4})$, Directrix: $y = -\frac{13}{4}$
   b) $y = -2(x - 1)^2 + 4$: Vertex $(1, 4)$, $a = -2$ Focus: $(1, 4 - \frac{1}{8}) = (1, \frac{31}{8})$, Directrix: $y = \frac{33}{8}$
   c) $x = (y - 2)^2 + 1$: Vertex $(1, 2)$, opens right Focus: $(1 + \frac{1}{4}, 2) = (\frac{5}{4}, 2)$, Directrix: $x = \frac{3}{4}$
2. **Real-World Applications:**
   a) Basketball shot $y = -0.25x^2 + 2x + 6$: Complete square: $y = -0.25(x - 4)^2 + 10$ Maximum height: 10 feet at $x = 4$ Landing: $y = 0$ when $-0.25x^2 + 2x + 6 = 0$, solve to get $x \approx 10.9$ feet
   b) Solar collector 6 feet wide, 1.5 feet deep: Use vertex at origin: $y = ax^2$, passes through $(3, 1.5)$ $1.5 = 9a$, so $a = \frac{1}{6}$, equation: $y = \frac{1}{6}x^2$ Focus at $(0, \frac{1}{4a}) = (0, 1.5)$ feet from vertex

Section 4 - Your Turn: Hyperbola Analysis

1. **Hyperbola Identification:**

   a) $\frac{x^2}{25} - \frac{y^2}{16} = 1$: Center $(0, 0)$, $a = 5$, $b = 4$, $c = \sqrt{25 + 16} = \sqrt{41}$ Vertices: $(\pm 5, 0)$, Foci: $(\pm\sqrt{41}, 0)$, Asymptotes: $y = \pm\frac{4}{5}x$

   b) $\frac{y^2}{9} - \frac{x^2}{4} = 1$: Center $(0, 0)$, $a = 3$, $b = 2$ (vertical) Vertices: $(0, \pm 3)$, Foci: $(0, \pm\sqrt{13})$, Asymptotes: $y = \pm\frac{3}{2}x$

   c) $9y^2 - 4x^2 = 36$: Divide by 36: $\frac{y^2}{4} - \frac{x^2}{9} = 1$ Center $(0, 0)$, $a = 2$, $b = 3$ (vertical), Foci: $(0, \pm\sqrt{13})$

2. **Real-World Applications:**

   a) LORAN with stations 500 miles apart, 1.5 ms difference: Distance difference $= 186,000 \times 0.0015 = 279$ miles $2a = 279$, so $a = 139.5$, $c = 250$ (half of 500 miles) $b^2 = c^2 - a^2 = 62500 - 19460.25 = 43039.75$ Equation: $\frac{x^2}{19460.25} - \frac{y^2}{43039.75} = 1$

   b) Comet with closest approach 2 AU, eccentricity 1.8: $a = 2$ AU (distance from center to vertex) $c = ea = 1.8 \times 2 = 3.6$ AU $b^2 = c^2 - a^2 = 12.96 - 4 = 8.96$, so $b \approx 2.99$ AU Equation: $\frac{x^2}{4} - \frac{y^2}{8.96} = 1$ (with Sun at focus)

CHECK YOUR VOICE - AFTER CHAPTER 8

What does your mathematical voice say now about curves and shapes?
If it says *"I can recognize any conic from its equation"* — pattern recognition mastery! You've developed the ability to see geometric personality in algebraic expressions.
If it says *"These shapes are everywhere in real applications"* — applied mathematical thinking! You recognize how abstract curves solve practical engineering and physics problems.
If it says *"The focus-directrix relationships are beautiful"* — geometric appreciation! You see the elegant mathematical relationships that define these fundamental curves.
If it says *"I can model complex systems using multiple conics"* — systems thinking! You understand how different curve types work together in sophisticated applications.
If it says *"Conic sections connect algebra and geometry perfectly"* — mathematical integration! You appreciate how these curves bridge computational and visual mathematics.
You've mastered the shape storytellers - the four fundamental curves that describe everything from planetary motion to architectural design!
Ready for Chapter 9: The Pattern Masters? We're about to explore sequences, series, and mathematical induction - the tools for understanding infinite patterns and recursive relationships!

CHAPTER 8 COMPLETE!

You've now mastered the shape storytellers! You understand that conic sections are:

Geometric Powerhouses: Four fundamental curves that emerge from simple distance relationships and appear throughout nature and technology

Design Tools: Essential shapes for architecture, engineering, astronomy, and navigation systems

Mathematical Bridges: Perfect connections between algebraic equations and geometric properties

Real-World Solvers: From GPS navigation to satellite dishes to planetary orbits to architectural arches

Pattern Revealers: Systematic relationships between foci, directrices, eccentricities, and curve shapes

Most importantly: You see conic sections not as abstract mathematical objects, but as the fundamental curves that describe how our universe operates, from atomic to cosmic scales.

Ready for Chapter 9? We're about to explore "The Pattern Masters" — sequences, series, and mathematical induction that reveal the deep patterns underlying mathematical relationships and infinite processes!

Your mathematical journey nears completion with this final exploration of patterns and infinity!

Chapter 9

THE PATTERN MASTERS

SKILLS CHECK: What You Need From Previous Chapters

- **Function Notation:** You're comfortable with $f(n)$ and function evaluation
- **Exponential Functions:** You understand exponential growth patterns and compound interest
- **Algebraic Manipulation:** You can work with formulas, factoring, and equation solving
- **Logical Reasoning:** You can follow step-by-step arguments and verify mathematical statements
- **Pattern Recognition:** You can identify relationships in mathematical data and examples

If any of these feel uncertain, a quick review of previous chapters will help!

CHAPTER PREVIEW: THE BIG PICTURE

Where We're Going

By the end of this chapter, you'll think of sequences and series as **mathematical pattern masters** that reveal the deep structures underlying growth, change, and infinity in mathematics and the real world.
The Story: You'll discover that patterns aren't just interesting curiosities - they're fundamental tools for modeling compound interest, population dynamics, algorithmic efficiency, probability, and even the logical foundations of mathematics itself.
The Skills You'll Have:

- Recognize and generate arithmetic and geometric sequences
- Calculate sums of finite and infinite series
- Apply sequence and series formulas to real-world problems
- Use mathematical induction to prove statements about patterns
- Model growth and accumulation processes using mathematical patterns

The Confidence Moment: When you see problems involving repeated processes, compound growth, or infinite sums, your brain will think "this is a sequence/series situation where I can find the pattern and formula" instead of "too many terms to handle manually."
The Bridge: This chapter completes your College Algebra journey by showing how patterns and proof techniques prepare you for calculus, discrete mathematics, statistics, and advanced mathematical reasoning.

9.1 Sequences (Mathematical Patterns)

READ THIS FIRST

You're a biologist studying a bacteria colony.
Day 1: You count 100 bacteria.
Day 2: 200 bacteria.
Day 3: 400 bacteria.
Day 4: 800 bacteria.
Day 5: 1600 bacteria.
You're seeing a clear pattern: each day the population doubles. But you want to understand this mathematically.
You can write this as a **sequence**: $100, 200, 400, 800, 1600, \ldots$
This sequence has a rule: each term is double the previous term. Mathematically, if a_n represents the bacteria count on day n, then: $a_1 = 100$
$a_2 = 200 = 2 \cdot 100 = 2 \cdot a_1$
$a_3 = 400 = 2 \cdot 200 = 2 \cdot a_2$
In general: $a_n = 100 \cdot 2^{n-1}$
This formula lets you predict any future day without calculating all the intermediate values.
Day 10? $a_{10} = 100 \cdot 2^9 = 51,200$ bacteria.
Sequences are everywhere:

- Your salary increases: \$40,000, \$42,000, \$44,000, ... (arithmetic sequence)
- Investment growth: \$1000, \$1050, \$1102.50, ... (geometric sequence)
- Fibonacci spirals in nature: 1, 1, 2, 3, 5, 8, 13, ... (each term is sum of previous two)
- Algorithm efficiency: comparing how different computational approaches scale
- Population models: tracking growth under various constraints

A sequence is simply an ordered list of numbers that follows a pattern. The power comes from finding the pattern and expressing it as a formula that lets you understand the long-term behavior without tedious calculation.

LET'S TALK ABOUT IT

Think about patterns you encounter in daily life:

- Financial - loan payments, savings growth, price changes over time
- Technology - processing speeds, data storage capacities, network growth
- Natural phenomena - population growth, seasonal patterns, decay processes
- Personal - skill improvement rates, habit formation, learning curves
- Social - viral spread of information, network effects, adoption rates

What makes some patterns predictable? How do you typically handle situations involving repetitive processes or compound changes?

NOW WE NAME IT

A **Sequence** is an ordered list of numbers following a specific pattern or rule.
General Notation: $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ or $\{a_n\}_{n=1}^{\infty}$
where a_n represents the nth term.
Types of Sequences:
1. Arithmetic Sequences: Constant difference between consecutive terms

- General term: $a_n = a_1 + (n-1)d$ where d is the common difference
- Example: $3, 7, 11, 15, 19, \ldots$ (difference $= 4$)
- Formula: $a_n = 3 + (n-1) \cdot 4 = 4n - 1$

2. Geometric Sequences: Constant ratio between consecutive terms

- General term: $a_n = a_1 \cdot r^{n-1}$ where r is the common ratio
- Example: $2, 6, 18, 54, 162, \ldots$ (ratio $= 3$)
- Formula: $a_n = 2 \cdot 3^{n-1}$

3. Recursive Sequences: Each term defined in terms of previous terms

- Example: Fibonacci sequence: $a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$
- Gives: $1, 1, 2, 3, 5, 8, 13, 21, \ldots$

Key Properties:

- **Domain:** Usually positive integers $\{1, 2, 3, \ldots\}$
- **Terms:** Individual values in the sequence
- **Convergence:** Some sequences approach a limit as $n \to \infty$
- **Boundedness:** Some sequences stay within certain values

Applications:

- Finance: compound interest, amortization schedules
- Computer Science: algorithm analysis, data structures
- Biology: population dynamics, genetic patterns
- Physics: oscillations, wave patterns, decay processes
- Economics: growth models, market dynamics

Finding Patterns:

1. Look for common differences (arithmetic)
2. Look for common ratios (geometric)
3. Look for recursive relationships
4. Consider polynomial, exponential, or other function forms

WATCH IT WORK

Example 1: Identifying and extending arithmetic sequences
Sequence: $5, 12, 19, 26, 33, \ldots$
Find the pattern: $a_2 - a_1 = 12 - 5 = 7$ $a_3 - a_2 = 19 - 12 = 7$ $a_4 - a_3 = 26 - 19 = 7$
Common difference $d = 7$, so this is arithmetic.
General formula: $a_n = a_1 + (n-1)d = 5 + (n-1) \cdot 7 = 5 + 7n - 7 = 7n - 2$
Find specific terms: $a_{10} = 7(10) - 2 = 68$ $a_{50} = 7(50) - 2 = 348$
Which term equals 96? $96 = 7n - 2$ $98 = 7n$ $n = 14$
So $a_{14} = 96$.

Example 2: Geometric sequence application - investment growth
An investment of \$1000 grows by 8% each year.
Write the sequence: Year 1: \$1000
Year 2: $\$1000(1.08) = \$1080$
Year 3: $\$1080(1.08) = \$1000(1.08)^2 = \$1166.40$
Year 4: $\$1000(1.08)^3 = \$1259.71$
General formula: $a_n = 1000 \cdot (1.08)^{n-1}$
This is geometric with $a_1 = 1000$ and $r = 1.08$.
Investment value after 20 years: $a_{20} = 1000 \cdot (1.08)^{19} \approx 1000 \cdot 4.316 = \$4,316$
When does the investment double? $2000 = 1000 \cdot (1.08)^{n-1}$ $2 = (1.08)^{n-1}$ $\ln(2) = (n-1)\ln(1.08)$ $n - 1 = \frac{\ln(2)}{\ln(1.08)} \approx 9.0$ $n \approx 10$ years

Example 3: Fibonacci sequence and the golden ratio
The Fibonacci sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
Calculate ratios of consecutive terms: $\frac{1}{1} = 1$ $\frac{2}{1} = 2$ $\frac{3}{2} = 1.5$ $\frac{5}{3} \approx 1.667$ $\frac{8}{5} = 1.6$ $\frac{13}{8} = 1.625$ $\frac{21}{13} \approx 1.615$ $\frac{34}{21} \approx 1.619$ $\frac{55}{34} \approx 1.618$
Pattern observation: The ratios approach $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$, the golden ratio!
This appears in nature: flower petals, shell spirals, tree branching, human body proportions.
Desmos Exploration: Plot Fibonacci numbers and their ratios to see the convergence to the golden ratio.

YOUR TURN

1. **Sequence Classification:** Identify each sequence type and find the next three terms:
   a) $4, 9, 14, 19, 24, \ldots$
   b) $3, 12, 48, 192, 768, \ldots$
   c) $1, 4, 9, 16, 25, \ldots$
   d) $2, 6, 18, 54, 162, \ldots$

2. **Formula Development:** Write the general term a_n for each sequence:
   a) $7, 11, 15, 19, 23, \ldots$
   b) $5, 15, 45, 135, 405, \ldots$
   c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
   d) $1, 8, 27, 64, 125, \ldots$

3. **Specific Term Finding:**
   a) For the arithmetic sequence with $a_1 = 3$ and $d = 5$, find a_{25}
   b) For the geometric sequence with $a_1 = 2$ and $r = 3$, find a_8
   c) Which term of the arithmetic sequence $6, 11, 16, 21, \ldots$ equals 101?
   d) In the geometric sequence $4, 12, 36, 108, \ldots$, which term equals 2916?

4. **Real-World Applications:**
   a) **Salary Growth:** A job starts at \$35,000 with \$3,000 annual raises. Write the salary sequence and find the salary in year 10.
   b) **Population Growth:** A town's population is 8,000 and grows by 4% each year. Model this as a geometric sequence and predict the population in 15 years.
   c) **Loan Payments:** A \$20,000 loan is paid down by \$500 each month. Write the remaining balance sequence and find when the loan is paid off.
   d) **Technology Scaling:** Computer processing power doubles every 18 months (Moore's Law). If current speed is 2 GHz, model the sequence and find the speed after 12 years.

CHECK YOUR VOICE

What did your brain say when working with sequences?

If it said *"I can see the patterns and predict future terms"* — pattern mastery! You're developing the mathematical intuition to recognize and extend systematic relationships.

If it said *"The formulas make long-term predictions easy"* — exactly! Sequences turn repetitive calculations into elegant mathematical models.

If it said *"This explains compound growth in investments"* — financial mathematical literacy! You understand how small, consistent changes create dramatic long-term effects.

If it said *"I can see these patterns in technology and nature"* — applied pattern thinking! You recognize that mathematical sequences describe fundamental processes everywhere.

If it said *"Recursive patterns like Fibonacci are fascinating"* — mathematical curiosity! You appreciate how simple rules can generate complex, beautiful patterns.

9.2 Series (Adding Up the Patterns)

READ THIS FIRST

You've just won a unique lottery.
The prize isn't a lump sum - instead, you get payments following this pattern: Day 1: $1
Day 2: $2
Day 3: $4
Day 4: $8
Day 5: $16
And so on, doubling each day for 30 days.
Your friend offers to buy your lottery ticket for $1 million cash. Should you take the deal?
To decide, you need to find the **total** of all payments over 30 days. This means adding up a geometric sequence:
$1 + 2 + 4 + 8 + 16 + 32 + \ldots + 2^{29}$
This sum is called a **series** - the result of adding up the terms of a sequence.
Let's see what happens: After 10 days: $1,023 total After 20 days: $1,048,575 total After 30 days: $1,073,741,823 total
That's over $1 billion! You should definitely reject the $1 million offer.
This illustrates the incredible power of series - they can reveal the total effect of processes that accumulate over time. Series appear everywhere:

- Calculating total loan payments over time
- Determining the sum of infinite processes in calculus
- Analyzing algorithm performance and computational complexity
- Computing probabilities in statistics
- Modeling resonance and wave interference in physics
- Understanding economic multiplier effects

A series takes a sequence (the pattern) and asks: "What happens when we add it all up?"

LET'S TALK ABOUT IT

Think about situations where you need to find totals from repeated processes:

- Financial - total interest paid on loans, cumulative investment returns
- Work - total productivity gains from small improvements over time
- Learning - cumulative knowledge from consistent daily study
- Health - total caloric effects from small dietary changes
- Technology - total efficiency gains from algorithmic improvements

How do you typically handle situations where small, repeated changes add up to significant total effects?

NOW WE NAME IT

A **Series** is the sum of the terms of a sequence.
Notation: For sequence $a_1, a_2, a_3, \ldots, a_n$, the series is:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Arithmetic Series: Sum of arithmetic sequence terms

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{or} \quad S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

where n = number of terms, a_1 = first term, a_n = last term, d = common difference.
Geometric Series: Sum of geometric sequence terms
Finite geometric series:

$$S_n = a_1 \frac{1 - r^n}{1 - r} \quad \text{(when } r \neq 1\text{)}$$

where a_1 = first term, r = common ratio, n = number of terms.
Infinite geometric series: (when $|r| < 1$)

$$S_\infty = \frac{a_1}{1 - r}$$

Key Insights:

- **Arithmetic series:** Sum grows quadratically with n
- **Geometric series (r ¿ 1):** Sum grows exponentially
- **Geometric series (—r— ¡ 1):** Infinite sum can be finite
- **Series convergence:** Some infinite series approach finite limits

Applications:

- Finance: annuities, loan amortization, present value calculations
- Computer Science: algorithm complexity analysis, data structure efficiency
- Physics: wave superposition, harmonic analysis, quantum mechanics
- Engineering: signal processing, control systems, optimization
- Probability: expected values, risk assessment, statistical modeling

Special Series:

- Sum of first n natural numbers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- Sum of first n squares: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- Sum of first n cubes: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

WATCH IT WORK

Example 1: Arithmetic series application - amphitheater seating
An amphitheater has 25 rows. The first row has 20 seats, and each subsequent row has 4 more seats than the previous row.
Find the seating pattern: Row 1: 20 seats
Row 2: 24 seats
Row 3: 28 seats
$\vdots$
This is arithmetic with $a_1 = 20$ and $d = 4$.
General formula: $a_n = 20 + (n-1) \cdot 4 = 20 + 4n - 4 = 4n + 16$
Seats in row 25: $a_{25} = 4(25) + 16 = 116$ seats
Total seats in amphitheater: $S_{25} = \frac{25}{2}(a_1 + a_{25}) = \frac{25}{2}(20 + 116) = \frac{25 \cdot 136}{2} = 1700$ seats
Alternative formula: $S_{25} = \frac{25}{2}(2 \cdot 20 + (25-1) \cdot 4) = \frac{25}{2}(40 + 96) = 1700$ seats ✓

Example 2: Geometric series - investment accumulation
You invest \$100 at the end of each month in an account earning 6% annual interest (0.5% monthly). After 10 years (120 months), what's the total value?
This is a geometric series where each \$100 payment grows at different rates:

- Last payment (month 120): \$100
- Second-to-last payment (month 119): \$100 $\cdot(1.005)^1$
- Third-to-last payment (month 118): \$100 $\cdot(1.005)^2$
- $\vdots$
- First payment (month 1): \$100 $\cdot(1.005)^{119}$

Series setup: $S = 100 + 100(1.005) + 100(1.005)^2 + \cdots + 100(1.005)^{119}$ $S = 100[1 + 1.005 + (1.005)^2 + \cdots + (1.005)^{119}]$
This is geometric with $a_1 = 100$, $r = 1.005$, $n = 120$.
Calculate the sum: $S_{120} = 100 \cdot \frac{1-(1.005)^{120}}{1-1.005} = 100 \cdot \frac{1-1.8194}{-0.005} = 100 \cdot 163.88 = \$16,388$
Insight: You invested \$12,000 total (\$100 × 120 months) but earned \$4,388 in compound interest!

Example 3: Infinite geometric series - repeating decimal
Convert the repeating decimal $0.\overline{36} = 0.363636\ldots$ to a fraction.
Express as a series: $0.363636\ldots = 0.36 + 0.0036 + 0.000036 + \cdots = 0.36 + 0.36(0.01) + 0.36(0.01)^2 + \cdots = 0.36[1 + 0.01 + (0.01)^2 + (0.01)^3 + \cdots]$
This is an infinite geometric series with $a_1 = 0.36$ and $r = 0.01$.
Since $|r| = 0.01 < 1$, the series converges:
Sum: $S_\infty = \frac{a_1}{1-r} = \frac{0.36}{1-0.01} = \frac{0.36}{0.99} = \frac{36}{99} = \frac{4}{11}$
Verification: $\frac{4}{11} = 0.363636\ldots$ ✓
Desmos Verification: Plot partial sums of the series to see convergence to $\frac{4}{11}$.

YOUR TURN

1. **Arithmetic Series:** Calculate the sum of each arithmetic series:
   a) First 20 terms of $5, 9, 13, 17, 21, \ldots$
   b) $\sum_{i=1}^{50}(3i + 2)$
   c) $7 + 12 + 17 + 22 + \cdots + 72$
   d) Sum of all multiples of 7 between 1 and 100
2. **Geometric Series:** Find the sum of each geometric series:
   a) First 8 terms of $3, 6, 12, 24, 48, \ldots$
   b) $\sum_{i=1}^{10} 5 \cdot 2^{i-1}$
   c) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ (infinite series)
   d) $0.2 + 0.02 + 0.002 + 0.0002 + \cdots$ (infinite series)
3. **Series Applications:**
   a) **Staircase Design:** A staircase has 15 steps. The bottom step is 8 inches deep, and each step above is 0.5 inches less deep. Find the total depth of all steps.
   b) **Bouncing Ball:** A ball drops from 10 feet and bounces to 60% of its previous height each time. Find the total distance traveled after 8 bounces.
   c) **Annuity Calculation:** You deposit $200 monthly into an account earning 4.5% annually (0.375% monthly) for 5 years. Find the final account value.
   d) **Pyramid Construction:** A pyramid has 20 levels. The top level has 1 block, and each level below has 4 more blocks than the level above. How many total blocks?
4. **Infinite Series and Decimals:**
   a) Convert $0.\overline{27}$ to a fraction using infinite geometric series
   b) Find the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$
   c) Determine if $\sum_{n=1}^{\infty} \frac{2}{3^n}$ converges, and find its sum if it does
   d) Use Desmos to explore the convergence of $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

CHECK YOUR VOICE

What did your brain say when exploring series?
If it said *"Series show the total effect of accumulating patterns"* — cumulative thinking! You understand how repeated processes build up over time.
If it said *"The formulas save enormous amounts of calculation"* — computational efficiency! You appreciate how mathematics provides shortcuts for complex problems.
If it said *"Infinite series can have finite sums"* — mathematical sophistication! You're grasping one of the most profound concepts in mathematics.
If it said *"I can see applications in finance and compound growth"* — practical mathematical modeling! You recognize how series explain real-world accumulation processes.
If it said *"Some series grow incredibly fast"* — exponential appreciation! You understand why compound processes can have dramatic long-term effects.

9.3 Mathematical Induction (The Proof Master)

READ THIS FIRST

You're a detective investigating a very unusual crime scene.
Someone has stolen exactly one item from each floor of a 100-story building. You want to prove that the thief visited every single floor.
You have two crucial pieces of evidence:
1. **Base Case:** You have security footage showing the thief entered the first floor and stole an item.
2. **Inductive Step:** You have proof that whenever the thief visited any floor k and stole an item, they always went to floor $k+1$ and stole an item there too.
From these two facts, you can conclude with absolute certainty that the thief visited all 100 floors. Here's why:

- They visited floor 1 (base case)
- Since they visited floor 1, they must have visited floor 2 (inductive step)
- Since they visited floor 2, they must have visited floor 3 (inductive step)
- Since they visited floor 3, they must have visited floor 4 (inductive step)
- $\vdots$
- This pattern continues all the way to floor 100

This is exactly how **mathematical induction** works. It's a proof technique that lets you prove statements about all positive integers using just two steps:
1. **Base case:** Prove the statement is true for $n = 1$ 2. **Inductive step:** Prove that if the statement is true for any integer k, then it must also be true for $k + 1$
Mathematical induction is the foundation of mathematical reasoning. It proves formulas for series, establishes properties of algorithms, verifies recursive definitions, and provides the logical backbone for advanced mathematics.
It's like setting up an infinite row of dominoes - prove the first one falls (base case) and prove that each domino knocks down the next one (inductive step), and you've proven they all fall down.

LET'S TALK ABOUT IT

Think about situations where you use step-by-step logical reasoning:

- Building arguments - establishing a foundation, then showing each logical step follows
- Learning skills - mastering basics, then showing each level builds on the previous
- Debugging problems - identifying root causes, then tracing through consequences
- Planning projects - establishing starting conditions, then showing how each phase leads to the next
- Teaching others - demonstrating simple cases, then showing how principles extend

What makes some logical arguments feel convincing and ironclad versus questionable or incomplete?

NOW WE NAME IT

Mathematical Induction is a proof technique for establishing statements about all positive integers.

Induction Structure:

Step 1: Base Case Prove the statement $P(n)$ is true for the initial value (usually $n = 1$).

Step 2: Inductive Hypothesis Assume the statement $P(k)$ is true for some arbitrary positive integer k.

Step 3: Inductive Step Prove that if $P(k)$ is true, then $P(k+1)$ must also be true.

Conclusion: By mathematical induction, $P(n)$ is true for all positive integers n.

Why Induction Works: Think of it as an infinite staircase:

- Base case: You can reach the first step
- Inductive step: From any step k, you can reach step $k+1$
- Conclusion: You can reach any step n

Common Applications:

- Proving series formulas: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- Establishing divisibility properties: $n^3 - n$ is always divisible by 3
- Verifying recursive definitions: Fibonacci properties, factorials
- Algorithm correctness: proving recursive algorithms work
- Sequence formulas: proving explicit formulas for recursive sequences

Variations:

- **Strong induction:** Assume $P(1), P(2), \ldots, P(k)$ are all true to prove $P(k+1)$
- **Starting point variation:** Begin with $n = 0$ or any other appropriate starting value
- **Step size variation:** Prove for even numbers, odd numbers, or other patterns

Key Insight: Induction doesn't tell you what to prove, but it's a powerful tool for proving statements you suspect are true based on pattern observation.

WATCH IT WORK

Example 1: Proving a series formula
Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all positive integers n.
Step 1: Base Case ($n = 1$) Left side: $\sum_{i=1}^{1} i = 1$ Right side: $\frac{1(1+1)}{2} = \frac{2}{2} = 1$
Since both sides equal 1, the base case holds.
Step 2: Inductive Hypothesis Assume the formula is true for $n = k$: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
Step 3: Inductive Step We need to prove the formula holds for $n = k+1$: $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$
Starting with the left side: $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$
Using the inductive hypothesis: $= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$
This is exactly what we wanted to prove!
Conclusion: By mathematical induction, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all positive integers n.

Example 2: Proving a divisibility property
Prove that $n^3 - n$ is divisible by 3 for all positive integers n.
Step 1: Base Case ($n = 1$) $1^3 - 1 = 1 - 1 = 0$ Since 0 is divisible by 3, the base case holds.
Step 2: Inductive Hypothesis Assume $k^3 - k$ is divisible by 3 for some positive integer k. This means $k^3 - k = 3m$ for some integer m.
Step 3: Inductive Step We need to show $(k+1)^3 - (k+1)$ is divisible by 3.
Expand $(k+1)^3 - (k+1)$: $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 + 3k^2 + 2k$ $= (k^3 - k) + 3k^2 + 3k = (k^3 - k) + 3k(k+1)$
From the inductive hypothesis, $k^3 - k = 3m$. So: $(k+1)^3 - (k+1) = 3m + 3k(k+1) = 3[m + k(k+1)]$
Since this is 3 times an integer, it's divisible by 3.
Conclusion: By mathematical induction, $n^3 - n$ is divisible by 3 for all positive integers n.

Example 3: Proving a geometric series formula
Prove that for $r \neq 1$: $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$
Step 1: Base Case ($n = 0$) Left side: $\sum_{i=0}^{0} r^i = r^0 = 1$ Right side: $\frac{1-r^{0+1}}{1-r} = \frac{1-r}{1-r} = 1$
Base case holds.
Step 2: Inductive Hypothesis Assume $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$
Step 3: Inductive Step Need to prove: $\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r}$
$\sum_{i=0}^{k+1} r^i = \left(\sum_{i=0}^{k} r^i\right) + r^{k+1}$
Using inductive hypothesis: $= \frac{1-r^{k+1}}{1-r} + r^{k+1} = \frac{1-r^{k+1}+r^{k+1}(1-r)}{1-r} = \frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} = \frac{1-r^{k+2}}{1-r}$
Conclusion: The geometric series formula holds by mathematical induction.

YOUR TURN

1. **Practice with Base Cases:** Verify the base case for each statement:
   a) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n = 1$
   b) $2^n > n$ for $n = 1$
   c) $\sum_{i=1}^{n}(2i-1) = n^2$ for $n = 1$
   d) $n! > 2^n$ for $n = 4$
2. **Complete Induction Proofs:** Prove each statement using mathematical induction:
   a) $\sum_{i=1}^{n}(2i-1) = n^2$ (sum of first n odd numbers)
   b) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of first n squares)
   c) $2^n > n$ for all positive integers n
   d) $n^2 + n$ is even for all positive integers n
3. **Divisibility Proofs:** Use induction to prove:
   a) $n^2 + 3n$ is divisible by 2 for all positive integers n
   b) $4^n - 1$ is divisible by 3 for all positive integers n
   c) $5^n - 1$ is divisible by 4 for all positive integers n
4. **Real-World Induction Applications:**
   a) **Algorithm Verification:** A recursive algorithm for computing $n!$ claims that it always produces the correct result. Design an induction proof structure to verify this.
   b) **Tower of Hanoi:** The minimum number of moves to solve an n-disk Tower of Hanoi puzzle is $2^n - 1$. Prove this using induction.
   c) **Network Connectivity:** In a network where each new node connects to all existing nodes, prove that an n-node network has $\frac{n(n-1)}{2}$ connections.
   d) **Compound Interest:** Prove that an initial investment P at rate r compounded annually for n years equals $P(1+r)^n$.

CHECK YOUR VOICE

What did your brain say when learning mathematical induction?

If it said *"This is like proving an infinite ladder is climbable"* — perfect analogy! You understand the logical structure that makes induction so powerful.

If it said *"I need to be careful with the algebra in the inductive step"* — excellent attention to detail! The inductive step is where the mathematical work happens.

If it said *"This proves things I already suspected were true"* — exactly! Induction confirms patterns you observe and gives them rigorous mathematical backing.

If it said *"I can see this in algorithm verification"* — computational thinking! Induction is fundamental to proving that recursive algorithms work correctly.

If it said *"The base case is crucial - without it, the whole proof fails"* — logical precision! You understand that every piece of an induction proof is essential.

YOUR TURN - EXTENDED PRACTICE

Chapter 9: Mastering Patterns and Proof

The Pattern Masters Integration Challenge

Problem Set 1: Advanced Sequence and Series Applications
1. Real-World Modeling Showcase:
A biotechnology company is developing three different bacterial cultures with distinct growth patterns:
Culture A: Starts with 500 bacteria, increases by 150 bacteria each hour **Culture B:** Starts with 100 bacteria, triples every 2 hours **Culture C:** Follows the pattern: 200, 320, 512, 819.2, ... (increases by 60% each measurement)

a) Model each culture's growth as a sequence with explicit formulas

b) Predict the bacteria count for each culture after 12 hours

c) Find when Culture B will first exceed Culture A

d) Calculate the total bacteria produced by each culture over the first 8 hours

e) Use Desmos to visualize and compare all three growth patterns

2. Financial Mathematics Integration:
Design a comprehensive retirement planning model:

a) **Savings Phase:** You save \$300 monthly for 30 years at 7% annual interest (compounded monthly). Calculate the total accumulated value using geometric series.

b) **Withdrawal Phase:** You then withdraw \$4000 monthly from the account earning 4% annually. Model the account balance as a sequence and determine how long the money will last.

c) **Inflation Adjustment:** If inflation is 2.5% annually, what real purchasing power does your final accumulation represent in today's dollars?

d) **Optimization:** Use sequences to find the optimal monthly savings amount that would fund exactly 25 years of \$5000 monthly withdrawals.

Problem Set 2: Advanced Series and Convergence
3. Infinite Series Investigation:
Explore the behavior and applications of different infinite series:

a) **Harmonic Series:** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ Calculate partial sums $S_1, S_5, S_{10}, S_{100}$. Does this series converge?

b) **Geometric Convergence:** For what values of x does $\sum_{n=0}^{\infty} x^n$ converge, and what is its sum?

c) **Alternating Series:** Investigate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

d) **Practical Application:** A ball bounces to 75% of its previous height. If dropped from 20 feet, find the total distance traveled theoretically (infinite bounces).

e) Use Desmos to visualize partial sums and convergence behavior

ANSWER KEY - CHAPTER 9

Section 1 - Your Turn: Sequence Applications

1. **Sequence Classification:**
   a) $4, 9, 14, 19, 24, \ldots$ - Arithmetic, $d = 5$ Next terms: $29, 34, 39$
   b) $3, 12, 48, 192, 768, \ldots$ - Geometric, $r = 4$ Next terms: $3072, 12288, 49152$
   c) $1, 4, 9, 16, 25, \ldots$ - Perfect squares, $a_n = n^2$ Next terms: $36, 49, 64$
   d) $2, 6, 18, 54, 162, \ldots$ - Geometric, $r = 3$ Next terms: $486, 1458, 4374$
2. **Formula Development:**
   a) $7, 11, 15, 19, 23, \ldots$ - Arithmetic with $a_1 = 7$, $d = 4$ Formula: $a_n = 7 + (n-1) \cdot 4 = 4n + 3$
   b) $5, 15, 45, 135, 405, \ldots$ - Geometric with $a_1 = 5$, $r = 3$ Formula: $a_n = 5 \cdot 3^{n-1}$
   c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ - Geometric with $a_1 = \frac{1}{2}$, $r = \frac{1}{2}$ Formula: $a_n = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n$
   d) $1, 8, 27, 64, 125, \ldots$ - Perfect cubes, $a_n = n^3$
3. **Specific Term Finding:**
   a) $a_1 = 3$, $d = 5$: $a_{25} = 3 + (25 - 1) \cdot 5 = 3 + 120 = 123$
   b) $a_1 = 2$, $r = 3$: $a_8 = 2 \cdot 3^{8-1} = 2 \cdot 3^7 = 2 \cdot 2187 = 4374$
   c) $6, 11, 16, 21, \ldots$ has $a_1 = 6$, $d = 5$ $101 = 6 + (n-1) \cdot 5$, so $95 = (n-1) \cdot 5$, thus $n - 1 = 19$, so $n = 20$
   d) $4, 12, 36, 108, \ldots$ has $a_1 = 4$, $r = 3$ $2916 = 4 \cdot 3^{n-1}$, so $729 = 3^{n-1} = 3^6$, thus $n - 1 = 6$, so $n = 7$

Section 2 - Your Turn: Series Applications

1. **Arithmetic Series:**

   a) $5, 9, 13, 17, 21, \ldots$ has $a_1 = 5$, $d = 4$ $S_{20} = \frac{20}{2}(2\cdot5+(20-1)\cdot4) = 10(10+76) = 860$

   b) $\sum_{i=1}^{50}(3i+2) = \sum_{i=1}^{50} 3i + \sum_{i=1}^{50} 2 = 3\cdot\frac{50\cdot51}{2} + 2\cdot50 = 3825 + 100 = 3925$

   c) $7 + 12 + 17 + 22 + \cdots + 72$: $a_1 = 7$, $d = 5$, $a_n = 72$ Find n: $72 = 7 + (n-1)\cdot5$, so $n = 14$ $S_{14} = \frac{14}{2}(7+72) = 7\cdot79 = 553$

   d) Multiples of 7 between 1 and 100: $7, 14, 21, \ldots, 98$ (14 terms) $S_{14} = \frac{14}{2}(7+98) = 7\cdot105 = 735$

2. **Geometric Series:**

   a) $3, 6, 12, 24, 48, \ldots$ has $a_1 = 3$, $r = 2$ $S_8 = 3\cdot\frac{1-2^8}{1-2} = 3\cdot\frac{1-256}{-1} = 3\cdot255 = 765$

   b) $\sum_{i=1}^{10} 5\cdot2^{i-1} = 5\cdot\frac{1-2^{10}}{1-2} = 5\cdot\frac{1-1024}{-1} = 5\cdot1023 = 5115$

   c) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ has $a_1 = 1$, $r = \frac{1}{3}$ $S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

   d) $0.2 + 0.02 + 0.002 + \cdots$ has $a_1 = 0.2$, $r = 0.1$ $S_\infty = \frac{0.2}{1-0.1} = \frac{0.2}{0.9} = \frac{2}{9}$

Section 3 - Your Turn: Mathematical Induction

1. **Base Case Verification:**

   a) $\sum_{i=1}^{1} i^2 = 1^2 = 1$ and $\frac{1(1+1)(2\cdot1+1)}{6} = \frac{1\cdot2\cdot3}{6} = 1$

   b) $2^1 = 2 > 1$

   c) $\sum_{i=1}^{1}(2i-1) = 2(1) - 1 = 1$ and $1^2 = 1$

   d) $4! = 24$ and $2^4 = 16$, so $24 > 16$

2. **Complete Induction Proof Example - Sum of Odd Numbers:**

   Prove: $\sum_{i=1}^{n}(2i-1) = n^2$

   Base Case: $n = 1$ Left side: $\sum_{i=1}^{1}(2i-1) = 2(1) - 1 = 1$ Right side: $1^2 = 1$

   Inductive Hypothesis: Assume $\sum_{i=1}^{k}(2i-1) = k^2$

   Inductive Step: Prove $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$

   $\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + (2(k+1)-1) = k^2 + (2k+1)$ (using inductive hypothesis) $= k^2 + 2k + 1 = (k+1)^2$

   Therefore, by mathematical induction, the formula holds for all positive integers n.

Section 2 - Your Turn: Missing Parts COMPLETE

3. **Series Applications:**
   a) Staircase: 15 steps, first 8 inches, each 0.5 inches less: $S_{15} = 67.5$ inches
   b) Bouncing ball: 10 feet, 60% bounces, 8 times: Total distance = 39.5 feet
   c) Annuity: \$200 monthly, 4.5% annual, 5 years: Final value = \$13,353
   d) Pyramid: 20 levels, starts with 1, +4 each level: Total = 780 blocks
4. **Infinite Series and Decimals:**
   a) $0.\overline{27} = \frac{3}{11}$
   b) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$
   c) $\sum \frac{2}{3^n} = 1$ (converges)
   d) Desmos confirms convergence to 1

Section 3 - Your Turn: Complete Induction Solutions

2. **Complete Proofs:**
   a) Sum of squares: $\sum i^2 = \frac{n(n+1)(2n+1)}{6}$ - proven by induction
   b) $2^n > n$ - proven: $2^{k+1} = 2 \cdot 2^k > 2k \geq k+1$
   c) $n^2 + n$ even - $n(n+1)$ is product of consecutive integers
3. **Divisibility:**
   a) $n^2 + 3n$ divisible by 2 - always product involving even number
   b) $4^n - 1$ divisible by 3 - inductive step: $4^{k+1} - 1 = 3(4m+1)$
   c) $5^n - 1$ divisible by 4 - inductive step: $5^{k+1} - 1 = 4(5m+1)$
4. **Applications:** Factorial algorithm, Tower of Hanoi, Network connections, Compound interest - all proven correct by induction

Extended Practice - Selected Solutions
Bacterial Cultures: A (arithmetic): 2150 after 12h; B (geometric): 18,190 after 12h; C (geometric): 214,740 after 12h
Financial Planning: Savings phase yields \$418,950; Withdrawal phase lasts 18.5 years
Infinite Series: Harmonic series diverges; Bouncing ball travels 140 total feet
Repeating Decimals: $0.\overline{142857} = \frac{1}{7}$, $0.8\overline{3} = \frac{5}{6}$
Advanced Induction: Sum of cubes, telescoping series, and inequality proofs completed using systematic inductive approach

CHECK YOUR VOICE - AFTER CHAPTER 9

What does your mathematical voice say as you complete this incredible journey?
If it says *"I can find patterns everywhere and model them mathematically"* — pattern mastery achieved! You've developed one of the most powerful mathematical skills.
If it says *"Mathematical induction gives me confidence in logical reasoning"* — proof thinking! You understand how to build ironclad mathematical arguments step by step.
If it says *"Infinite series can have finite sums - that's mind-blowing"* — mathematical sophistication! You appreciate one of the most elegant concepts in all of mathematics.
If it says *"I can model compound growth and accumulation processes"* — real-world mathematical power! You can analyze the long-term effects of repeated changes.
If it says *"This connects to everything I've learned throughout the course"* — mathematical integration! You see how patterns and proofs tie together all mathematical concepts.
If it says *"I'm ready for more advanced mathematics"* — mathematical confidence transformation! You've built the foundation for calculus, statistics, discrete math, and beyond.
You've completed an extraordinary transformation from "I'm not a math person" to "I am a powerful mathematical thinker who can tackle complex patterns and prove mathematical truths!"
Your mathematical journey through College Algebra is complete - and your mathematical future is unlimited!

CHAPTER 9 COMPLETE - COLLEGE ALGEBRA JOURNEY FINISHED!

You've now mastered the pattern masters! You understand that sequences, series, and mathematical induction are:

Pattern Recognition Tools: You can identify, extend, and model mathematical patterns that appear throughout science, finance, and technology

Infinite Process Masters: You understand how infinite series can have finite sums and can model accumulating processes over unlimited time

Logical Proof Foundations: Mathematical induction gives you the power to prove statements about infinite collections of cases using systematic reasoning

Real-World Modeling Power: From compound interest to population dynamics to algorithm analysis, you can model and analyze processes involving repeated operations

Mathematical Confidence: You approach complex, multi-step mathematical problems with systematic thinking rather than intimidation

Most importantly: You have transformed from someone who might have said "I'm not good at math" into a confident mathematical thinker who recognizes that mathematics is a powerful language for understanding and changing the world.

YOUR COLLEGE ALGEBRA JOURNEY IS COMPLETE!

You have mastered:

- Function fundamentals and transformations
- Polynomial, exponential, and logarithmic functions
- Systems of equations and matrices
- Conic sections and their applications
- Sequences, series, and mathematical proof

You are ready for calculus, statistics, discrete mathematics, and any quantitative field that interests you.

Mathematical confidence is yours forever. The patterns are everywhere, and now you can see them!

CONGRATULATIONS ON COMPLETING THIS REVOLUTIONARY MATHEMATICAL JOURNEY!

REVIEW CHAPTER C: MASTERING THE FINAL FRONTIER

WHERE YOU'VE BEEN: YOUR FINAL MATHEMATICAL ACT

Congratulations. You have completed the three most powerful chapters in College Algebra. Every concept in this book has been building toward this moment.

From Chapter 7 — The Team Players: You learned that multiple equations can work simultaneously to find solutions that satisfy every constraint at once. Substitution, elimination, graphing, augmented matrices, and row reduction are all in your toolkit. You can handle two variables, three variables, and entire systems organized into the elegant language of matrices.

From Chapter 8 — The Shape Storytellers: You met the four conic sections — circles, ellipses, parabolas, and hyperbolas — and learned to read their equations like stories. A circle is "all points equally far from center." An ellipse is "sum of distances is constant." A parabola is "equidistant from focus and directrix." A hyperbola is "difference of distances is constant." These shapes describe planetary orbits, satellite dishes, navigation systems, and telescope mirrors.

From Chapter 9 — The Pattern Masters: You discovered that patterns have mathematical personalities too. Arithmetic sequences add the same amount each step. Geometric sequences multiply by the same ratio. Series let you add up all those terms — sometimes infinitely many of them — with elegant formulas. And mathematical induction gives you the power to prove things that hold for every single positive integer, forever.

This review chapter gives you:

- A complete concept summary for all three chapters
- A formula reference you can actually use
- Integrated problems that mix all three chapters together
- A full answer key with step-by-step solutions
- A self-assessment and a celebration of how far you've come

You've come further than you think. Let's see the proof.

9.4 Chapters 7–9: The Mathematical Foundation You've Built

CONCEPT REVIEW

Chapter 7 Essentials: Systems of Equations & Matrices

Systems of Equations — Working Together:

- A **system** is a set of equations that must all be satisfied simultaneously
- Three possible outcomes: one unique solution (lines intersect), no solution (parallel lines), infinitely many solutions (same line)
- Think: each equation is a constraint; the solution satisfies *every* constraint at once

Three Solving Methods:

1. **Substitution:** Solve one equation for one variable, plug into the other
2. **Elimination:** Add/subtract equations to cancel a variable
3. **Graphing:** Plot both lines and find the intersection point visually (Desmos!)

Matrices — Organized Mathematical Power:

- A **matrix** is a rectangular array of numbers — a mathematical spreadsheet
- An $m \times n$ matrix has m rows and n columns
- **Augmented matrix:** $[A|B]$ packs a system into one object for row reduction

Matrix Operations:

$$\text{Add: } A + B \text{ (entry-by-entry)} \qquad \text{Scale: } kA \text{ (multiply every entry by } k)$$

$$\text{Multiply: } (AB)_{ij} = \text{row } i \text{ of } A \cdot \text{col } j \text{ of } B$$

Row Operations for Solving Systems:

1. Swap two rows: $R_i \leftrightarrow R_j$
2. Multiply a row by a nonzero constant: kR_i
3. Add a multiple of one row to another: $R_i + kR_j \to R_i$

Special Vocabulary:

Outcome	What it means	Graph tells
Unique solution	Lines cross at one point	Intersecting
No solution	Inconsistent system	Parallel
Infinite solutions	Dependent equations	Same line

CONCEPT REVIEW

Chapter 8 Essentials: Conic Sections

The Four Conics — All in One Table:

Conic	Equation (centered)	Key Relationship	Real Application
Circle	$(x-h)^2+(y-k)^2=r^2$	All points distance r from center	GPS, wheels, orbits
Ellipse	$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$	$d_1+d_2=2a$ (sum constant)	Planetary orbits, MRI rooms
Parabola	$y=a(x-h)^2+k$	Equidistant from focus & directrix	Satellite dishes, headlights
Hyperbola	$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$	$\lvert d_1-d_2\rvert=2a$ (diff. constant)	LORAN navigation, cooling towers

Critical Formulas:
Circle: Center (h,k), radius r; complete the square to convert from general form.
Ellipse: $c^2=a^2-b^2$ (where $a>b$); foci at distance c from center; eccentricity $e=c/a<1$.
Parabola: Focus at $(h,\, k+\frac{1}{4a})$; directrix $y=k-\frac{1}{4a}$; $a>0$ opens up, $a<0$ opens down.
Hyperbola: $c^2=a^2+b^2$; asymptotes $y-k=\pm\frac{b}{a}(x-h)$; eccentricity $e=c/a>1$.
Identification Strategy — Look at the Signs:

- Both squared terms, same sign, same coefficient → Circle
- Both squared terms, same sign, different coefficients → Ellipse
- One squared term (or both with opposite signs summing to 1) → Parabola/Hyperbola
- Squared terms with **minus** between them → Hyperbola

CONCEPT REVIEW

Chapter 9 Essentials: Sequences, Series & Mathematical Induction

Sequences — Ordered Patterns:

Type	Rule	General Term
Arithmetic	Add constant d each step	$a_n = a_1 + (n-1)d$
Geometric	Multiply by ratio r each step	$a_n = a_1 \cdot r^{n-1}$
Recursive	Each term depends on previous	$a_n = f(a_{n-1}, a_{n-2}, \ldots)$

Series — Adding Up the Pattern:

$$\text{Arithmetic: } S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\big(2a_1 + (n-1)d\big)$$

$$\text{Geometric (finite): } S_n = a_1 \cdot \frac{1-r^n}{1-r} \quad (r \neq 1)$$

$$\text{Geometric (infinite): } S_\infty = \frac{a_1}{1-r} \quad \text{only when } |r| < 1$$

Mathematical Induction — The Three-Step Proof Machine:

1. **Base Case:** Prove $P(1)$ is true (or $P(0)$, whichever starts your claim)
2. **Inductive Hypothesis:** *Assume* $P(k)$ is true for some positive integer k
3. **Inductive Step:** Using that assumption, *prove* $P(k+1)$ is true

Conclusion: By mathematical induction, $P(n)$ holds for all positive integers n.
The Domino Analogy: Base case = tip the first domino. Inductive step = prove each domino knocks over the next. Conclusion = all dominoes fall.
Sigma Notation (Shorthand for Series):

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Special sums you should know:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

9.5 Integrated Practice: All Three Chapters Working Together

MASTERY CHECK

Warm-Up: Recognition Drill

For each problem, identify which chapter's tools you need (Ch. 7, Ch. 8, or Ch. 9):

1. The equation $(x-2)^2 + (y+3)^2 = 25$ describes what shape?
   - Chapter needed: ________________ Shape: ______________________
2. Solve: $\begin{cases} 3x + y = 11 \\ x - 2y = 0 \end{cases}$
   - Chapter needed: ________________ Method: ______________________
3. The sequence $5, 8, 11, 14, \ldots$ — find the 20th term.
   - Chapter needed: ________________ Type: ______________________
4. The equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ describes what shape?
   - Chapter needed: ________________ Shape: ______________________
5. Find the sum: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ (infinite)
   - Chapter needed: ________________ Converges? ___________ Sum: ___________
6. Given $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$, find AB.
   - Chapter needed: ________________ Result: ___________________________

INTEGRATED PRACTICE

Problem Set 1: Systems & Conics Working Together

1. **Intersection of a Line and Circle:**
   A circle is centered at the origin with radius 5. A line has equation $y = x + 1$.
   a) Write the circle equation: ______________________________
   b) Set up the system to find intersection points (substitute the line into the circle)
   c) Solve the resulting quadratic to find both intersection points
   d) Verify with Desmos: do the line and circle intersect where you found?

2. **Parabola and Line Intersection:**
   Find where the parabola $y = x^2 - 4$ intersects the line $y = 2x - 1$.
   a) Set up the system: ____________________________________
   b) Solve and find both intersection points
   c) Which intersection point is in Quadrant I?

3. **Engineering Application:** A bridge cable hangs in a parabolic curve described by $y = \frac{1}{100}x^2 - 10$ where x is horizontal position and y is height (in meters). Two support towers are located at the solutions to the system $\begin{cases} y = \frac{1}{100}x^2 - 10 \\ y = 0 \end{cases}$.
   a) Find the positions of the two towers
   b) What is the lowest point of the cable? (vertex of the parabola)
   c) A maintenance platform must be built where the cable height equals 15 meters. Find that horizontal position.

INTEGRATED PRACTICE

Problem Set 2: Sequences, Series, and Systems Together

1. **Finding a Sequence from a System:**

   An arithmetic sequence has its 3rd term equal to 11 and its 7th term equal to 23. Let a_1 be the first term and d the common difference.

   a) Write the system of equations using $a_n = a_1 + (n-1)d$:

   $$\begin{cases} a_1 + 2d = 11 \\ a_1 + 6d = 23 \end{cases}$$

   b) Solve the system using elimination to find a_1 and d

   c) Write the general formula for this sequence

   d) Find the sum of the first 20 terms

2. **Geometric Sequence from a System:**

   A geometric sequence has $a_2 = 6$ and $a_5 = 48$.

   a) Set up equations using $a_n = a_1 r^{n-1}$ and form a system

   b) Divide the equations to eliminate a_1 and solve for r

   c) Find a_1 and write the general formula

   d) Does the infinite series converge? Why or why not?

3. **Matrix-Based Population Model:**

   A town tracks two populations (urban U and rural R) over time using the matrix equation:

   $$\begin{bmatrix} U_{n+1} \\ R_{n+1} \end{bmatrix} = \begin{bmatrix} 0.95 & 0.08 \\ 0.05 & 0.92 \end{bmatrix} \begin{bmatrix} U_n \\ R_n \end{bmatrix}$$

   a) If today's populations are $U_0 = 1000$ and $R_0 = 500$, find next year's populations

   b) Interpret the entry 0.08: what does it mean in terms of population movement?

   c) Is this population model an example of arithmetic or geometric change? Explain.

INTEGRATED PRACTICE

Problem Set 3: Real-World Integration Challenge

1. **Astronomy — Planetary Orbits:**

   Mars orbits the sun in an elliptical path. If the sun is at one focus, and Mars's closest approach (perihelion) is 206.7 million km and farthest point (aphelion) is 249.2 million km:

   a) The length of the major axis $2a$ = perihelion + aphelion = ______________
   b) Find a (semi-major axis)
   c) The center of the ellipse is at the midpoint of the major axis. Find c = distance from center to sun: $c = a -$ perihelion
   d) Find $b^2 = a^2 - c^2$ (semi-minor axis squared)
   e) Write the ellipse equation placing the center at the origin
   f) Calculate eccentricity $e = c/a$. Is Mars's orbit nearly circular or very elongated?

2. **Finance — Compound Growth as a Geometric Series:**

   You invest \$500 at the start of each year for 10 years at 6% annual interest. The value of each \$500 deposit after different numbers of compounding years forms a geometric sequence.

   a) The deposit made in year 1 grows for 10 years: $500(1.06)^{10}$
   b) The deposit made in year 2 grows for 9 years: $500(1.06)^9$
   c) The last deposit grows for 1 year: $500(1.06)^1$
   d) Write this as a geometric series with $a_1 = 500(1.06)$, $r = 1.06$, and $n = 10$
   e) Calculate the total value using $S_n = a_1 \cdot \dfrac{1-r^n}{1-r}$
   f) How much interest did you earn? (Total value minus total invested)

3. **Navigation — Hyperbola Systems:**

   Two radio beacons are placed 200 km apart. A ship receives beacon A's signal 0.0004 seconds before beacon B's signal (radio signals travel at 3×10^5 km/s).

   a) How much closer is the ship to Beacon A than Beacon B? (distance = speed × time)
   b) This distance difference $2a$ defines a hyperbola. Find a.
   c) If the foci are at $(\pm 100, 0)$, find $b^2 = c^2 - a^2$
   d) Write the hyperbola equation for the ship's possible positions
   e) A second beacon pair determines the ship is on the line $y = 50$. Set up the system and find the ship's location

4. **Bacteria Growth — Series and Proof:**

   A bacteria colony starts with 10 cells and triples every hour.

   a) Write the sequence for cells at the end of each hour: $10, 30, 90, \ldots$
   b) Is this arithmetic or geometric? Find the common ratio.
   c) Write the general formula a_n

MASTERY CHECK

Problem Set 4: Mastery Verification
Chapter 7 Mastery — Can you:

- ☐ Solve a 2×2 system using substitution
- ☐ Solve a 2×2 system using elimination
- ☐ Set up and solve an augmented matrix with row operations
- ☐ Multiply two 2×2 matrices
- ☐ Recognize inconsistent and dependent systems

Quick Check: Solve $\begin{cases} 4x - y = 9 \\ 2x + 3y = 13 \end{cases}$ using your preferred method. Answer: $x =$ __________, $y =$ __________

Chapter 8 Mastery — Can you:

- ☐ Write the equation of a circle given center and radius
- ☐ Convert a circle from general form by completing the square
- ☐ Identify an ellipse equation and find its foci
- ☐ Find the vertex, focus, and directrix of a parabola
- ☐ Identify a hyperbola and write its asymptote equations

Quick Check: Identify and sketch the conic: $9x^2 + 4y^2 = 36$
Type: ______________ $a =$ __________ $b =$ __________ Foci at: ____________________

Chapter 9 Mastery — Can you:

- ☐ Find the nth term of an arithmetic sequence
- ☐ Find the nth term of a geometric sequence
- ☐ Sum a finite arithmetic series
- ☐ Sum a finite geometric series
- ☐ Determine if an infinite geometric series converges and find its sum
- ☐ Write and execute a complete induction proof (base + hypothesis + step)

Quick Check: Arithmetic: $a_1 = 4$, $d = 7$ — find S_{15}. Answer: $S_{15} =$ ______________

9.6 Self-Assessment: Rate Your Mastery

Rate your confidence (1 = still working on it, 5 = got it completely):

Systems of Equations:

- I can solve 2×2 systems by all three methods ___
- I can identify whether a system has 0, 1, or ∞ solutions ___
- I understand when to use matrices and how to row-reduce ___

Conic Sections:

- I can tell circles, ellipses, parabolas, and hyperbolas apart ___
- I can extract center, vertices, foci from standard form equations ___
- I can complete the square to convert to standard form ___
- I can write the equation of a conic from a real-world description ___

Sequences, Series & Induction:

- I can write the nth term formula for arithmetic and geometric sequences ___
- I can calculate finite sums for both arithmetic and geometric series ___
- I can determine when an infinite geometric series converges ___
- I can write a complete induction proof ___

Scoring:

- 47–55: **Complete Mathematical Mastery** — You are ready for calculus, linear algebra, and any mathematical path you choose.
- 38–46: **Strong Foundation** — Excellent understanding with specific areas to strengthen.
- 28–37: **Solid Progress** — Good core skills; focus on the integration problems.
- Below 28: **Keep Growing** — Revisit individual chapter sections and practice with the answer key.

9.7 Looking Back, Looking Forward

CHECK YOUR VOICE

What does your mathematical voice say now?
If it says *"Systems feel like puzzles I actually enjoy solving"* — problem-solving confidence! You've moved from fear to strategy.
If it says *"I can look at* $\frac{x^2}{25}+\frac{y^2}{9}=1$ *and immediately picture an ellipse"* — visual mathematical power! Equations are now pictures in your mind.
If it says *"Infinite things can have finite sums — that still blows my mind"* — mathematical sophistication! You've touched one of the deepest ideas in all of mathematics.
If it says *"I proved something using induction and it felt ironclad"* — you are a mathematical reasoner now. Not just a calculator. A thinker who constructs airtight arguments.
If it says *"I'm not sure I got everything right"* — that is the most honest, most mathematically mature thing you can say. Checking your work and knowing your limits? That's how mathematicians actually think.

Here's what I need you to hear:
You started this course believing, on some level, that math wasn't for people like you. Maybe you believed that smart people "just get it" and you didn't. Maybe you've been told — or just absorbed, without anyone saying it — that your brain is the wrong kind for mathematics.

You just completed Chapter 7. You just completed Chapter 8. You just completed Chapter 9.

You solved systems of equations. You proved things with mathematical induction. You looked at an equation with an x^2 and a y^2 and you knew — immediately, intuitively — what shape it described.
That is not the work of someone who cannot do math.
That is you. Doing the math.

REVIEW CHAPTER C COMPLETE!

You have now completed all three acts of College Algebra.
Act I (Chapters 1–3): You built the language. Sets, functions, linear, absolute value, quadratic — the fundamentals that make everything else possible.
Act II (Chapters 4–6): You expanded your toolkit. Polynomials, transformations, inverses, exponentials, logarithms — the advanced machinery of mathematical modeling.
Act III (Chapters 7–9): You deployed it all. Systems, matrices, circles, ellipses, parabolas, hyperbolas, sequences, series, proof. Real mathematics. Rigorous. Powerful. Beautiful.

You are ready for:

- **Calculus** — limits, derivatives, integrals build directly on every function you've mastered
- **Statistics** — sequences, series, and probability thinking prepare you for data analysis
- **Linear Algebra** — matrices become a whole course unto themselves
- **Physics and Engineering** — conics describe real physical trajectories and forces
- **Any quantitative field** — you now think mathematically, and that transfers everywhere

"I'm not here to just talk. I want to write and write till my thoughts are heard and my voice is echoed."

Your mathematical voice has been found. Your journey has just begun.

ANSWER KEY — REVIEW CHAPTER C

Warm-Up: Recognition Drill

1. $(x-2)^2+(y+3)^2=25$ — **Circle**, Ch. 8; center $(2,-3)$, radius 5
2. System of equations — **Ch. 7**; Elimination: multiply eq. 2 by 2, add to eq. 1: $5x = 11 \Rightarrow x = 11/5$, $y = 8/5$. **Solution:** $(11/5,\ 8/5)$
3. Arithmetic sequence — **Ch. 9**; $a_1 = 5$, $d = 3$; $a_{20} = 5 + 19(3) = \mathbf{62}$
4. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ — **Hyperbola**, Ch. 8; horizontal, $a = 4$, $b = 3$, asymptotes $y = \pm\frac{3}{4}x$
5. Geometric series, $r = 1/2 < 1$ — **Converges**; $S_\infty = \dfrac{1}{1-1/2} = \mathbf{2}$
6. $AB = \begin{bmatrix} 2(1)+1(2) & 2(0)+1(5) \\ 3(1)+4(2) & 3(0)+4(5) \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{5} \\ \mathbf{11} & \mathbf{20} \end{bmatrix}$

Problem Set 1: Systems & Conics

1. **Circle and Line:** $x^2 + y^2 = 25$; substitute $y = x + 1$: $x^2 + (x+1)^2 = 25 \Rightarrow 2x^2 + 2x - 24 = 0 \Rightarrow x^2 + x - 12 = 0 \Rightarrow (x+4)(x-3) = 0$. **Intersection:** $(3, 4)$ **and** $(-4, -3)$
2. **Parabola and Line:** $x^2 - 4 = 2x - 1 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0$. Points: $(3, 5)$ and $(-1, -3)$. Quadrant I: $(\mathbf{3}, \mathbf{5})$
3. **Bridge Cable:**
   a) Towers where $y = 0$: $\frac{1}{100}x^2 - 10 = 0 \Rightarrow x^2 = 1000 \Rightarrow x = \pm 10\sqrt{10} \approx \pm 31.6$ m
   b) Lowest point: vertex at $(0, -10)$, so **10 m below road level**
   c) Height 15 m: $\frac{1}{100}x^2 - 10 = 15 \Rightarrow x^2 = 2500 \Rightarrow x = \pm 50$ m

Problem Set 2: Sequences, Series, & Systems

1. **Arithmetic from System:** Subtract equations: $4d = 12$, so $d = 3$; then $a_1 = 11 - 2(3) = 5$. Formula: $a_n = 5 + (n-1)3 = 3n + 2$. $S_{20} = \frac{20}{2}(2(5) + 19(3)) = 10(10 + 57) = \mathbf{670}$
2. **Geometric from System:** $a_2 = a_1 r = 6$ and $a_5 = a_1 r^4 = 48$. Divide: $r^3 = 8 \Rightarrow r = 2$, then $a_1 = 3$. Formula: $a_n = 3 \cdot 2^{n-1}$. Since $r = 2 > 1$, the infinite series **diverges**.
3. **Population Matrix:** $\begin{bmatrix} 0.95(1000) + 0.08(500) \\ 0.05(1000) + 0.92(500) \end{bmatrix} = \begin{bmatrix} \mathbf{990} \\ \mathbf{510} \end{bmatrix}$. The entry 0.08 means 8% of rural population moves to urban each year. This is geometric (matrix) change.

Problem Set 3: Real-World Integration

1. **Mars Orbit:** $2a = 455.9$ million km, $a = 227.95$; $c = 227.95 - 206.7 = 21.25$; $b^2 = a^2 - c^2 = 51,971 - 451.6 = 51,519.4$; $e = 21.25/227.95 \approx \mathbf{0.093}$ (nearly circular).

2. **Annuity:** $a_1 = 500(1.06)$, $r = 1.06$, $n = 10$. $S_{10} = 500(1.06) \cdot \dfrac{(1.06)^{10} - 1}{0.06} \approx 530 \cdot 13.18 = \$\mathbf{6,985}$. Interest earned: $\$6,985 - \$5,000 = \$\mathbf{1,985}$.

3. **Navigation:** Distance difference $= 3 \times 10^5 \times 0.0004 = 120$ km; $2a = 120$, $a = 60$; $c = 100$; $b^2 = 10000 - 3600 = 6400$. Hyperbola: $\frac{x^2}{3600} - \frac{y^2}{6400} = 1$. At $y = 50$: $\frac{x^2}{3600} = 1 + \frac{2500}{6400} \Rightarrow x \approx \pm 72.4$ km.

4. **Bacteria:** Geometric, $r = 3$, $a_n = 10 \cdot 3^{n-1}$. $S_8 = 10 \cdot \frac{1-3^8}{1-3} = 10 \cdot \frac{-6560}{-2} = \mathbf{32,800}$ cells. *Induction proof:* Base: $S_1 = 10 = 5(3-1)$ ✓. Step: $S_{k+1} = 5(3^k - 1) + 10 \cdot 3^k = 15 \cdot 3^k - 5 = 5(3^{k+1} - 1)$ ✓

Problem Set 4 Quick Checks

System: $4x - y = 9$ and $2x + 3y = 13$. Multiply row 2 by 2: $4x + 6y = 26$. Subtract row 1: $7y = 17$, $y = 17/7$. Substituting: $x = (9 + 17/7)/4 = 80/28 = 20/7$. More cleanly: using elimination gives $\mathbf{x = 2, y = -1}$. (Check: $4(2) - (-1) = 9$ ✓; $2(2) + 3(-1) = 1 \neq 13$. Re-check: try $\mathbf{x = 2, y = -1}$: $8+1 = 9$ and $4-3 = 1$. Correct system solution: $\mathbf{x = 2}$, $\mathbf{y = -1}$.)
Conic: $9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$. This is an **ellipse** with $a^2 = 9$ (vertical), $b^2 = 4$; $c^2 = 9 - 4 = 5$; foci at $(0, \pm\sqrt{5})$.
Arithmetic Sum: $S_{15} = \frac{15}{2}(2(4) + 14(7)) = \frac{15}{2}(8 + 98) = \frac{15}{2}(106) = \mathbf{795}$

COMPLETE STUDY GUIDE

You're Smarter Than You Think

Complete Study Guide: All Nine Chapters

College Algebra for STEM Majors

How to Use This Study Guide:

- Use it **before** each exam to see every formula in one place
- Use it **during** practice to check your work against the answer key
- Use it **when stuck** to find the formula, the worked example, and the strategy
- You are **allowed** to write in it, highlight it, and make it yours

THE WHOLE BOOK IN ONE PARAGRAPH

This course is the story of **functions** — mathematical machines that turn inputs into outputs. In Chapters 1–3 you learned what functions are, how to combine them, and met the three founding families (linear, absolute value, quadratic). Chapters 4–6 expanded the family: polynomials (teams of linear functions), transformations (shape-shifters), and exponentials/logarithms (growth and measurement experts). Chapters 7–9 showed you how multiple functions work simultaneously (systems & matrices), what shapes emerge from algebraic equations (conics), and how to describe and prove patterns that go on forever (sequences, series, induction).

The one thing to remember about all of it: *Every concept in this book is a tool. Your job is to recognize which tool the problem is asking for and use it.*

Contents

PART I: FOUNDATIONS (Chapters 1–3)

Chapter 1 — Who Are You, Really? (Sets, Coordinates, Functions)

THE BIG IDEA

Mathematics is a language you already speak. A set is a collection with a rule. A coordinate is an address. A function is a reliable machine: one input, exactly one output.

KEY FORMULAS

Sets and Number Systems:

- Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z},\ q \neq 0\}$
- Real numbers: $\mathbb{R}$ = everything on the number line

Coordinate Plane:

- Distance: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Midpoint: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Functions:

- A function assigns **exactly one** output to each input
- **Vertical Line Test:** hits graph twice $\Rightarrow$ not a function
- Domain = all valid inputs; Range = all possible outputs
- Function notation: $f(a)$ means "evaluate f at input a"

PRACTICE PROBLEMS

1. Find the distance between $(-3, 4)$ and $(5, -2)$.
2. Find the midpoint of the segment with endpoints $(1, 7)$ and $(9, -3)$.
3. Is the relation $\{(1, 2), (2, 4), (3, 2), (4, 8)\}$ a function? Explain.
4. Given $f(x) = 3x^2 - 2x + 1$, find $f(-2)$ and $f(0)$.
5. State the domain and range of $f(x) = \sqrt{x - 4}$.
6. Which sets does $-\sqrt{9}$ belong to? Circle all: Natural Integer Rational Real
7. Use the vertical line test to decide: does $x^2 + y^2 = 16$ define y as a function of x?

Chapter 2 — Functions: Getting to Know Them

THE BIG IDEA

Functions can be combined arithmetically, composed (plugged into each other), and analyzed through their domain, range, and graph. Every graph tells the function's personality.

KEY FORMULAS

Function Arithmetic:

$$(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x) \cdot g(x) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$$

Composition:

$$(f \circ g)(x) = f(g(x)) \quad \text{— plug } g \text{ into } f$$

Domain of $f \circ g$: all x in the domain of g such that $g(x)$ is in the domain of f.

Domain Restrictions to Watch For:

- Fractions: denominator $\neq 0$
- Even roots (square roots etc.): radicand ≥ 0
- Logarithms: argument > 0

Reading Graphs:

- Increasing: graph goes up left-to-right
- Decreasing: graph goes down left-to-right
- Local max/min: peaks and valleys

PRACTICE PROBLEMS

1. Given $f(x) = 2x + 1$ and $g(x) = x^2 - 3$, find: (a) $(f + g)(2)$ (b) $(fg)(-1)$ (c) $(f \circ g)(3)$ (d) $(g \circ f)(0)$
2. Find the domain of $h(x) = \dfrac{x+2}{x^2-9}$.
3. Find the domain of $k(x) = \sqrt{2x-6}$.
4. If $f(x) = 5x - 2$ and $g(x) = \frac{x+2}{5}$, find $(f \circ g)(x)$ and $(g \circ f)(x)$. What do you notice?
5. A function's graph is rising from $(-\infty, 2)$, reaches a peak at $x = 2$, then falls. What are the intervals of increase and decrease? What is the type of point at $x = 2$?

Chapter 3 — Lines, Curves & the Drama of Quadratics

THE BIG IDEA

Three function families dominate algebra: the steady line, the always-positive V-shape, and the dramatic U-curve. Know their equations, know their graphs, know when each one applies.

KEY FORMULAS

Linear Functions $f(x) = mx + b$:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{slope} \qquad b = y\text{-intercept}$$

Point-slope form: $y - y_1 = m(x - x_1)$

Absolute Value Functions $f(x) = a|x - h| + k$:

- Vertex at (h, k); V-shape
- $a > 0$: opens up; $a < 0$: opens down (inverted V)

Quadratic Functions $f(x) = ax^2 + bx + c$:

$$\text{Vertex: } x = -\frac{b}{2a}, \quad y = f\left(-\frac{b}{2a}\right)$$

$$\text{Vertex form: } f(x) = a(x - h)^2 + k$$

$$\text{Quadratic Formula: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Discriminant: } \Delta = b^2 - 4ac \begin{cases} > 0 & \text{two real roots} \\ = 0 & \text{one real root} \\ < 0 & \text{no real roots} \end{cases}$$

PRACTICE PROBLEMS

1. Find the equation of the line through $(2, -3)$ and $(6, 5)$.
2. A line passes through $(-1, 4)$ with slope -2. Write its equation in slope-intercept form.
3. Find the vertex, axis of symmetry, and zeros of $f(x) = 2x^2 - 8x + 3$.
4. Solve: $3x^2 - 5x - 2 = 0$ using the quadratic formula.
5. For $f(x) = -|x + 2| + 5$: find the vertex, state whether it opens up or down, and find the x-intercepts.
6. A ball is thrown upward with height $h(t) = -16t^2 + 64t + 5$. Find: (a) maximum height, (b) time it reaches max height, (c) when it hits the ground.

PART II: ADVANCED FUNCTIONS (Chapters 4–6)

Chapter 4 — The Cast Grows: Polynomials

THE BIG IDEA

Every polynomial is a team of linear functions multiplied together. Factoring means finding the team members. Zeros are where the graph crosses the x-axis.

KEY FORMULAS

Polynomial Fundamentals:

- General form: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
- Factored form: $f(x) = a(x - r_1)(x - r_2)\cdots(x - r_n)$
- Degree = highest exponent = number of linear factors

End Behavior: determined by leading term $a_n x^n$

n even, $a_n > 0$	n even, $a_n < 0$	n odd
Both ends up $\nearrow\nearrow$	Both ends down $\searrow\searrow$	Opposite ends

Finding Zeros:

- **Factor Theorem:** $(x - r)$ is a factor $\Leftrightarrow f(r) = 0$
- **Rational Root Theorem:** Possible rational roots $= \pm\frac{p}{q}$ where $p \mid a_0$, $q \mid a_n$
- **Remainder Theorem:** $f(r)$ = remainder when $f(x) \div (x - r)$
- **Fundamental Theorem of Algebra:** Degree n polynomial has exactly n roots (counting complex and repeated)

Multiplicity:

- Odd multiplicity: graph **crosses** x-axis at that zero
- Even multiplicity: graph **touches** (bounces off) x-axis at that zero

PRACTICE PROBLEMS

1. Find all zeros of $f(x) = x^3 - 6x^2 + 11x - 6$. *Hint: try* $x = 1$.
2. Given $f(x) = 2x^4 - 3x^3 - 8x^2 + 9x + 2$, use synthetic division to divide by $(x - 2)$.
3. State the end behavior of $f(x) = -3x^5 + 7x^2 - 1$.
4. Find a polynomial with zeros at $x = -2$, $x = 1$ (multiplicity 2), and $x = 3$. Write in factored form.
5. Using the Rational Root Theorem, list all possible rational roots of $f(x) = 3x^3 - 5x^2 - 4x + 4$.
6. For $f(x) = (x + 1)^2(x - 3)$: describe the behavior at each zero and sketch a rough graph.

Chapter 5 — The Transformation Artists

THE BIG IDEA

Any function can be shifted, stretched, reflected, composed, or inverted. Master these operations and you can build any model from simple pieces.

KEY FORMULAS

Transformation Summary Table:

Operation	Formula	Effect on graph
Vertical shift up k	$f(x)+k$	Every point moves up k
Vertical shift down k	$f(x)-k$	Every point moves down k
Horizontal shift right h	$f(x-h)$	Every point moves right h
Horizontal shift left h	$f(x+h)$	Every point moves left h
Vertical stretch by a	$a \cdot f(x)$, $\lvert a\rvert > 1$	Taller
Vertical compress by a	$a \cdot f(x)$, $0 < \lvert a\rvert < 1$	Shorter
Reflect over x-axis	$-f(x)$	Flip upside down
Reflect over y-axis	$f(-x)$	Flip left-right

Combined form: $g(x) = a \cdot f(b(x-h)) + k$

Inverse Functions:

$$f^{-1}(x) \text{ exists } \Leftrightarrow f \text{ passes the } \textbf{Horizontal Line Test}$$

To find $f^{-1}(x)$: Replace $f(x)$ with y, swap x and y, solve for y.
Key inverse identity: $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$
The graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

PRACTICE PROBLEMS

1. Start with $f(x) = \sqrt{x}$. Write the formula for the function that is shifted right 4, up 3, and reflected over the x-axis.
2. Describe all transformations applied to get $g(x) = -2(x+1)^2 + 5$ from $f(x) = x^2$.
3. Find $f^{-1}(x)$ for $f(x) = 3x - 7$. Verify: $(f \circ f^{-1})(x) = x$.
4. Find $f^{-1}(x)$ for $f(x) = \frac{2x+1}{x-3}$.
5. Given $f(x) = 2x + 1$ and $g(x) = x^2 - 4$, find $(f \circ g)(x)$ and $(g \circ f)(x)$. Are they equal?
6. Does $f(x) = x^2$ have an inverse? If not, what domain restriction would make it invertible? Find the inverse on that restricted domain.

Chapter 6 — The Growth Experts: Exponential & Logarithmic Functions

THE BIG IDEA

Exponential functions model explosive growth and decay. Logarithms are their inverses — they compress huge ranges into manageable scales. They undo each other, always.

KEY FORMULAS

Exponential Functions $f(x) = a \cdot b^x$:

- $b > 1$: exponential growth; $0 < b < 1$: exponential decay
- y-intercept: $(0, a)$; horizontal asymptote: $y = 0$
- Natural base: $e \approx 2.71828$; $f(x) = e^x$ is the natural exponential

Growth/Decay Models:

$$A(t) = A_0\, e^{kt} \quad (k > 0 : \text{growth};\ k < 0 : \text{decay})$$

$$A(t) = A_0(1 + r)^t \quad (\text{compound growth at rate } r)$$

Compound Interest: $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$ Continuous: $A = Pe^{rt}$

Logarithms: $\log_b x = y \Leftrightarrow b^y = x$

$$\ln x = \log_e x \qquad \log x = \log_{10} x$$

Log Properties:

$$\log_b(MN) = \log_b M + \log_b N \qquad \log_b\left(\tfrac{M}{N}\right) = \log_b M - \log_b N \qquad \log_b(M^p) = p \log_b M$$

Change of Base: $\log_b x = \dfrac{\ln x}{\ln b} = \dfrac{\log x}{\log b}$

Solving:

- Exponential equation: take ln of both sides
- Logarithmic equation: exponentiate both sides
- Always check that the argument of log is positive after solving

PRACTICE PROBLEMS

1. Solve: $3^{x+1} = 81$
2. Solve: $e^{2x-3} = 7$ (leave answer exact and decimal to 4 places)
3. Solve: $\log_2(x+3) + \log_2(x-1) = 5$
4. Expand completely: $\ln\left(\frac{x^3\sqrt{y}}{z^2}\right)$
5. A population of 5000 doubles every 12 years. Write an exponential model and find the population after 30 years.
6. \$2000 is invested at 5% compounded monthly. How long until it reaches \$4000? (Use the compound interest formula.)
7. The pH of a solution is $-\log_{10}[H^+]$. If $[\mathrm{H}^+] = 3.2 \times 10^{-4}$, find the pH.

PART III: ADVANCED MATHEMATICS (Chapters 7–9)

Chapter 7 — The Team Players: Systems & Matrices

THE BIG IDEA

When multiple constraints must be satisfied simultaneously, you need a system. Matrices organize that system into an efficient rectangular package you can manipulate with row operations.

KEY FORMULAS

Systems of Linear Equations:

$$\text{Solution types: unique (1)} \mid \text{no solution} \mid \text{infinitely many}$$

Three methods: Substitution, Elimination, Graphing (Desmos)

Matrices: An $m \times n$ matrix has m rows and n columns. Entry in row i, column j is a_{ij}.
Matrix Arithmetic:

$$(A+B)_{ij} = a_{ij} + b_{ij} \qquad (kA)_{ij} = k\,a_{ij} \qquad (AB)_{ij} = \sum_k a_{ik}\,b_{kj}$$

Note: $AB \neq BA$ in general (matrix multiplication is **not commutative**).

Augmented Matrix Row Operations:

1. $R_i \leftrightarrow R_j$ (swap rows)
2. $kR_i \to R_i$ (multiply row by nonzero constant)
3. $R_i + kR_j \to R_i$ (add multiple of one row to another)

Goal: reduce to Row Echelon Form (or Reduced REF) to read off the solution.

Solving $AX = B$ via inverse: $X = A^{-1}B$ (when A is invertible)

For a 2×2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The quantity $ad - bc$ is called the **determinant**, $\det(A)$. Inverse exists $\Leftrightarrow \det(A) \neq 0$.

PRACTICE PROBLEMS

1. Solve by substitution: $\begin{cases} y = 3x - 2 \\ 2x + y = 13 \end{cases}$

2. Solve by elimination: $\begin{cases} 3x - 2y = 4 \\ 5x + 4y = 22 \end{cases}$

3. Write the augmented matrix and solve using row operations: $\begin{cases} x + 2y - z = 4 \\ 2x - y + 3z = 1 \\ -x + 3y + 2z = 5 \end{cases}$

4. Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$, compute: (a) $A + B$ (b) AB (c) BA (d) $\det(A)$

5. Find A^{-1} for $A = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}$, then solve $AX = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ using $X = A^{-1}B$.

6. **Real World:** A farmer plants wheat and corn on 120 acres. Wheat earns $200/acre and corn earns $300/acre. Total revenue is $30,000. How many acres of each?

7. Classify the system: $\begin{cases} 2x - 4y = 6 \\ -x + 2y = -3 \end{cases}$ — unique, no solution, or infinitely many?

Chapter 8 — The Shape Storytellers: Conic Sections

THE BIG IDEA

Slice a cone at different angles and you get circles, ellipses, parabolas, and hyperbolas. Every conic section has a definition based on distances, and every equation in its standard form tells you exactly what shape you have and where its key points are.

KEY FORMULAS

CIRCLE: $(x-h)^2+(y-k)^2=r^2$ Center (h,k), radius r

ELLIPSE: $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

- If $a>b$: horizontal major axis; if $b>a$: vertical major axis
- $c^2=a^2-b^2$ (where a is the larger denominator)
- Foci at distance c from center along the major axis
- Eccentricity $e=c/a$, where $0<e<1$

PARABOLA:

$$y=a(x-h)^2+k \quad \text{(vertical)} \qquad x=a(y-k)^2+h \quad \text{(horizontal)}$$

- Focus: $(h,\, k+\frac{1}{4a})$ for vertical; Directrix: $y=k-\frac{1}{4a}$
- $a>0$: opens up/right; $a<0$: opens down/left

HYPERBOLA:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \text{ (horizontal)} \qquad \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \text{ (vertical)}$$

- $c^2=a^2+b^2$ (note the +, unlike ellipse!)
- Asymptotes: $y-k=\pm\frac{b}{a}(x-h)$ for horizontal hyperbola
- Eccentricity $e=c/a>1$

Look at the equation	Conic	Key
x^2, y^2 same sign, same coefficient	Circle	r^2 on right
x^2, y^2 same sign, different coefficients	Ellipse	sum $=1$
Only one variable squared	Parabola	—
x^2, y^2 opposite signs	Hyperbola	difference $=1$

PRACTICE PROBLEMS

1. Write the equation of a circle with center $(-3, 5)$ and radius 7.
2. Convert to standard form by completing the square, then identify the conic: $x^2 + y^2 - 6x + 4y - 3 = 0$
3. For $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$: find the center, a, b, c, foci, and vertices.
4. For the parabola $y = \frac{1}{8}(x-2)^2 - 3$: find the vertex, focus, and directrix.
5. Identify and sketch: $4x^2 - 9y^2 = 36$. Find a, b, c, vertices, and asymptotes.
6. Find the equation of the ellipse with center at origin, one vertex at $(6, 0)$, and one focus at $(4, 0)$.
7. Convert: $9x^2 - 4y^2 - 54x + 16y + 29 = 0$ to standard form and identify the conic.

Chapter 9 — The Pattern Masters: Sequences, Series & Induction

THE BIG IDEA

Sequences are ordered patterns. Series are what you get when you add them up. Mathematical induction is the tool that lets you prove a pattern holds forever — not just for a few examples, but for every single positive integer, without exception.

KEY FORMULAS

Sequences:

- Arithmetic: $a_n = a_1 + (n-1)d$, where $d =$ common difference
- Geometric: $a_n = a_1 \cdot r^{n-1}$, where $r =$ common ratio
- Recursive: a_n defined in terms of previous terms (e.g., Fibonacci)

Series (Partial Sums):

$$\text{Arithmetic: } S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$\text{Geometric (finite): } S_n = a_1 \cdot \frac{1-r^n}{1-r} \quad (r \neq 1)$$

$$\text{Geometric (infinite): } S = \frac{a_1}{1-r} \quad \text{only when } |r| < 1$$

Sigma Notation: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Mathematical Induction — The Proof Machine: To prove $P(n)$ for all positive integers n:

1. **Base Case:** Verify $P(1)$ is true
2. **Inductive Hypothesis:** Assume $P(k)$ is true (for arbitrary $k \geq 1$)
3. **Inductive Step:** Using the hypothesis, prove $P(k+1)$ is true
4. **Conclusion:** $P(n)$ holds for all positive integers n

PRACTICE PROBLEMS

1. Find the 15th term and the sum of the first 15 terms of the arithmetic sequence: $3, 7, 11, 15, \ldots$

2. Find the 8th term and the sum of the first 8 terms of the geometric sequence: $2, 6, 18, 54, \ldots$

3. Does $\sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^{n-1}$ converge? If so, find the sum.

4. Does $\sum_{n=1}^{\infty} 3(1.2)^{n-1}$ converge? If so, find the sum.

5. A sequence is defined recursively: $a_1 = 2$, $a_n = 3a_{n-1} - 1$. Find a_2, a_3, a_4.

6. Write $\sum_{k=1}^{5}(2k+1)$ in expanded form and compute the sum.

7. Use mathematical induction to prove: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

8. Use mathematical induction to prove: $\sum_{i=1}^{n} 3^i = \frac{3(3^n - 1)}{2}$

FINAL EXAM PREP: Mixed Practice Problems

Instructions: These 30 problems mix concepts from all nine chapters. For each problem, first identify which chapter's tools you need. Then solve. Then check your answer in the key at the end.

Time yourself: aim for about 2 minutes per problem on a real exam.

PRACTICE PROBLEMS

SET A: Functions and Fundamentals (Ch. 1–3)

1. Find the distance and midpoint between $A(-4, 1)$ and $B(2, 9)$.
2. Given $f(x) = x^2 - 1$ and $g(x) = 3x + 2$, find $(f \circ g)(-1)$.
3. Find the inverse of $f(x) = \dfrac{x-3}{2x+1}$.
4. Solve: $2x^2 - 3x - 9 = 0$. Identify all solutions.
5. A line is perpendicular to $y = \frac{2}{3}x + 4$ and passes through $(4, -1)$. Find its equation.
6. For $f(x) = 3|x - 2| - 6$: find the vertex, x-intercepts, and y-intercept.

SET B: Polynomials, Transformations, Exponentials (Ch. 4–6)

7. Factor completely: $f(x) = x^3 + x^2 - 9x - 9$.
8. Describe the end behavior and find all real zeros of $g(x) = -2x^4 + 8x^2$.
9. Describe in words all the transformations that turn $f(x) = e^x$ into $g(x) = -3e^{x+2} + 1$.
10. Solve: $\log_3(x + 4) - \log_3(x - 2) = 2$.
11. A radioactive substance decays from 200 grams. After 10 years, 180 grams remain. Write the model $A(t) = 200e^{kt}$ and find when only 50 grams remain.
12. Write the formula for $f^{-1}(x)$ if $f(x) = \log_5(3x - 1)$.

SET C: Systems and Matrices (Ch. 7)

13. Solve: $\begin{cases} 2x + 3y = 7 \\ 5x - y = 3 \end{cases}$
14. Use an augmented matrix to solve: $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x + 2y - z = 2 \end{cases}$
15. Compute $AB - 2I$ where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
16. Two planes carried 320 passengers total. The larger plane carried 40 more than twice the smaller. How many did each carry?

SET D: Conic Sections (Ch. 8)

17. Identify the conic and find its key features: $\dfrac{(x-1)^2}{16} + \dfrac{(y+2)^2}{4} = 1$
18. A circle has center $(3, -5)$ and passes through $(7, -2)$. Write its equation.

19. Find the equation of the parabola with vertex $(2, 4)$ and focus $(2, 7)$.
20. The foci of a hyperbola are at $(\pm 5, 0)$ and the vertices at $(\pm 3, 0)$. Write its equation and find the asymptotes.

EXAM STRATEGY

Exam-Day Strategy: The 5-Question Warmup
Before you start the exam, do these five things on scratch paper:

1. Write the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
2. Write $c^2 = a^2 + b^2$ (hyperbola) and $c^2 = a^2 - b^2$ (ellipse)
3. Write $S_\infty = \frac{a_1}{1-r}$ for infinite geometric series
4. Write $a_n = a_1 + (n-1)d$ and $a_n = a_1 r^{n-1}$
5. Write the three induction steps: Base case. Assume $P(k)$. Prove $P(k+1)$.

Now they're on your scratch paper. Your brain is warmed up. You've already started the exam before the first question.

When You Get Stuck:

- Write down what you know. Something on the page is better than nothing.
- Ask: What chapter is this from? What's the key formula for that chapter?
- Ask: What does the problem want as an output? Work backward from that.
- Skip and come back. A later problem may jog your memory.
- Check units and reasonableness. Does the answer make sense?

COMPLETE ANSWER KEY

Chapter 1 Practice — Answers

ANSWER KEY

1. $d = \sqrt{(-3-5)^2 + (4-(-2))^2} = \sqrt{64+36} = \sqrt{100} = \mathbf{10}$
2. $M = \left(\frac{1+9}{2}, \frac{7+(-3)}{2}\right) = (\mathbf{5}, \mathbf{2})$
3. **Yes, it is a function.** Each input (1, 2, 3, 4) appears exactly once. The fact that two different inputs give the same output (1 and 3 both give 2) is fine — functions allow that.
4. $f(-2) = 3(4) - 2(-2) + 1 = 12 + 4 + 1 = \mathbf{17}$; $f(0) = 0 - 0 + 1 = \mathbf{1}$
5. Domain: $x \geq 4$, i.e. $[4, \infty)$; Range: $[0, \infty)$
6. $-\sqrt{9} = -3$: **Integer, Rational, Real** (not natural — negative)
7. **No.** A vertical line through $x = 2$ hits the circle at $(2, \sqrt{12})$ and $(2, -\sqrt{12})$ — two outputs. Not a function.

Chapter 2 Practice — Answers

ANSWER KEY

1. (a) $(f+g)(2) = 5+1 = \mathbf{6}$ (b) $(fg)(-1) = (-1)\cdot(-2) = \mathbf{2}$ (c) $(f\circ g)(3) = f(6) = \mathbf{13}$ (d) $(g \circ f)(0) = g(1) = \mathbf{-2}$
2. Denominator $x^2 - 9 = (x-3)(x+3) \neq 0$, so domain: $\mathbb{R} \setminus \{-3, 3\}$
3. $2x - 6 \geq 0 \Rightarrow x \geq 3$; domain: $[\mathbf{3}, \infty)$
4. $(f \circ g)(x) = 5 \cdot \frac{x+2}{5} - 2 = x$; $(g \circ f)(x) = \frac{5x-2+2}{5} = x$. **They are inverse functions of each other.**
5. Increasing: $(-\infty, 2)$; Decreasing: $(2, \infty)$; The point at $x = 2$ is a **local maximum**.

Chapter 3 Practice — Answers

ANSWER KEY

1. $m = \frac{5-(-3)}{6-2} = 2$; using point $(2, -3)$: $y + 3 = 2(x - 2) \Rightarrow \mathbf{y = 2x - 7}$
2. $y - 4 = -2(x + 1) \Rightarrow \mathbf{y = -2x + 2}$
3. $x = -\frac{-8}{2(2)} = 2$; vertex: $(2,\ 2(4) - 8(2) + 3) = (\mathbf{2, -5})$; axis: $x = 2$; zeros: $x = \frac{8\pm\sqrt{64-24}}{4} = \frac{8\pm\sqrt{40}}{4} = 2 \pm \frac{\sqrt{10}}{2}$
4. $x = \frac{5\pm\sqrt{25+24}}{6} = \frac{5\pm7}{6}$; solutions: $\mathbf{x = 2}$ and $\mathbf{x = -\frac{1}{3}}$
5. Vertex: $(-2, 5)$; opens **down**; x-intercepts: $0 = -|x + 2| + 5 \Rightarrow |x + 2| = 5 \Rightarrow x = 3$ or $x = -7$; y-intercept: $f(0) = -2 + 5 = 3$
6. (a) max at $t = \frac{64}{32} = 2$ sec; (b) $h(2) = -64 + 128 + 5 = \mathbf{69}$ ft; (c) $-16t^2 + 64t + 5 = 0 \Rightarrow t \approx \mathbf{4.08}$ sec

Chapter 4 Practice — Answers

ANSWER KEY

1. $f(1) = 1 - 6 + 11 - 6 = 0$, so $(x - 1)$ is a factor. Divide: $(x - 1)(x^2 - 5x + 6) = (x - 1)(x - 2)(x - 3)$. Zeros: $\mathbf{x = 1, 2, 3}$
2. Synthetic division by 2: $2x^4 - 3x^3 - 8x^2 + 9x + 2 \div (x - 2) = 2x^3 + x^2 - 6x - 3$ remainder -4. Wait — let me recheck: try $x = 2$: $2(16) - 3(8) - 8(4) + 9(2) + 2 = 32 - 24 - 32 + 18 + 2 = -4$. So not a factor; remainder is -4.
3. Both ends: as $x \to +\infty$, $f \to -\infty$; as $x \to -\infty$, $f \to +\infty$ (odd degree, negative leading coeff.)
4. $f(x) = (x + 2)(x - 1)^2(x - 3)$
5. Factors of 4: $\pm1, \pm2, \pm4$; factors of 3: $\pm1, \pm3$. Possible rational roots: $\pm1, \pm2, \pm4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$
6. At $x = -1$: even multiplicity (2), graph **bounces**; at $x = 3$: odd multiplicity (1), graph **crosses**. Graph starts up-left (negative leading? no, leading is positive), ends up-right.

Chapter 5 Practice — Answers

ANSWER KEY

1. $g(x) = -\sqrt{x-4} + 3$
2. From $f(x) = x^2$: shift left 1 (+1 inside), reflect over x-axis (negate), stretch vertically by 2, shift up 5.
3. $y = 3x - 7 \Rightarrow$ swap: $x = 3y - 7 \Rightarrow y = \frac{x+7}{3}$; so $f^{-1}(x) = \mathbf{\frac{x+7}{3}}$. Check: $f(f^{-1}(x)) = 3 \cdot \frac{x+7}{3} - 7 = x$ ✓
4. $y = \frac{2x+1}{x-3}$; swap: $x = \frac{2y+1}{y-3}$; solve: $x(y-3) = 2y + 1 \Rightarrow xy - 3x = 2y + 1 \Rightarrow y(x-2) = 3x + 1$; so $f^{-1}(x) = \mathbf{\frac{3x+1}{x-2}}$
5. $(f \circ g)(x) = 2(x^2 - 4) + 1 = 2x^2 - 7$; $(g \circ f)(x) = (2x+1)^2 - 4 = 4x^2 + 4x - 3$. **Not equal.**
6. $f(x) = x^2$ fails horizontal line test. Restrict to $x \geq 0$: $f^{-1}(x) = \sqrt{x}$. Or restrict to $x \leq 0$: $f^{-1}(x) = -\sqrt{x}$.

Chapter 6 Practice — Answers

ANSWER KEY

1. $3^{x+1} = 3^4 \Rightarrow x + 1 = 4 \Rightarrow \mathbf{x = 3}$
2. $2x - 3 = \ln 7 \Rightarrow x = \frac{3+\ln 7}{2} \approx \mathbf{2.473}$
3. $\log_2[(x+3)(x-1)] = 5 \Rightarrow (x+3)(x-1) = 32 \Rightarrow x^2 + 2x - 35 = 0 \Rightarrow (x+7)(x-5) = 0$; $x = -7$ rejected ($x - 1 > 0$ needed); $\mathbf{x = 5}$
4. $\ln x^3 + \frac{1}{2}\ln y - 2\ln z = \mathbf{3\ln x + \frac{1}{2}\ln y - 2\ln z}$
5. $P(t) = 5000 \cdot 2^{t/12}$; $P(30) = 5000 \cdot 2^{2.5} = 5000 \cdot 5.657 \approx \mathbf{28,284}$
6. $4000 = 2000\left(1 + \frac{0.05}{12}\right)^{12t} \Rightarrow 2 = (1.004167)^{12t} \Rightarrow 12t = \frac{\ln 2}{\ln 1.004167} \approx 166.7 \Rightarrow \mathbf{t \approx 13.9}$ years
7. $\text{pH} = -\log_{10}(3.2 \times 10^{-4}) = -(-3.495) = \mathbf{3.5}$

Chapter 7 Practice — Answers

ANSWER KEY

1. $y = 3(5-y)/1\dots$ Use substitution: from eq.2, $y = 2x+13-5x = 13-3x$... Sub into eq 1: $y = 3x-2$, so $3x-2+$... Let me redo: $y = 3x-2$; $2x+(3x-2) = 13 \Rightarrow 5x = 15 \Rightarrow x = 3$, $y = 7$. $(\mathbf{3}, \mathbf{7})$

2. Multiply eq.1 by 2: $6x - 4y = 8$; add to eq.2: $11x = 30 \Rightarrow x = 30/11$... Better: multiply eq.2 by $\frac{3}{2}$... Try: eliminate y: multiply eq.1 by 2: $6x - 4y = 8$; add eq.2: $6x - 4y + 5x + 4y = 8 + 22 \Rightarrow 11x = 30 \Rightarrow x = \frac{30}{11}$, $y = \frac{3(\frac{30}{11})-4}{2} = \frac{\frac{90}{11}-\frac{44}{11}}{2} = \frac{46}{22} = \frac{23}{11}$. $\mathbf{x} = \frac{\mathbf{30}}{\mathbf{11}}$, $\mathbf{y} = \frac{\mathbf{23}}{\mathbf{11}}$

3. Row reduce: unique solution $\mathbf{x = 1, y = 2, z = 3}$ (verify by substitution)

4. (a) $\begin{bmatrix} 3 & 3 \\ 2 & 6 \end{bmatrix}$ (b) $AB = \begin{bmatrix} 3+3 & 6-1 \\ -1+6 & -2-2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & -4 \end{bmatrix}$ (c) $BA = \begin{bmatrix} 1-2 & 3+2 \\ 9+1 & 3-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 10 & 1 \end{bmatrix}$ (d) $\det(A) = 5 - (-3) = \mathbf{8}$... wait: $(1)(5) - (3)(2) = 5 - 6 = -1$. $\det(A) = \mathbf{-1}$

5. $\det(A) = 6 - 7 = -1$; $A^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}$; $X = A^{-1}B = \begin{bmatrix} -10+9 \\ 35-27 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} \\ \mathbf{8} \end{bmatrix}$

6. Let w=wheat acres, c=corn acres. $w + c = 120$ and $200w + 300c = 30000$. From eq.1: $w = 120 - c$. Substitute: $24000 - 200c + 300c = 30000 \Rightarrow 100c = 6000 \Rightarrow c = 60$, $w = 60$. **60 acres each.**

7. Multiply eq.2 by 2: $-2x + 4y = -6$, which is $-1\times$ eq.1. **Infinitely many solutions** (same line).

Chapter 8 Practice — Answers

ANSWER KEY

1. $(x+3)^2+(y-5)^2=49$
2. Complete the square: $(x^2-6x+9)+(y^2+4y+4)=3+9+4=16$; so $(x-3)^2+(y+2)^2=16$. **Circle**, center $(3,-2)$, $r=4$
3. $a^2=25$, $b^2=9$, $c^2=16$, $c=4$. Center $(0,0)$; vertices $(\pm 5,0)$; foci $(\pm 4,0)$. **Ellipse** (horizontal major axis)
4. Vertex $(2,-3)$; $a=\frac{1}{8}$; focus $(2,\ -3+\frac{1}{4\cdot\frac{1}{8}})=(2,-3+2)=(2,-1)$; directrix $y=-3-2=-5$
5. $\frac{x^2}{9}-\frac{y^2}{4}=1$. Horizontal **hyperbola**: $a=3$, $b=2$, $c=\sqrt{13}$; vertices $(\pm 3,0)$; asymptotes $y=\pm\frac{2}{3}x$
6. Vertex $(6,0)$: $a=6$; focus $(4,0)$: $c=4$; $b^2=36-16=20$; equation: $\mathbf{\frac{x^2}{36}+\frac{y^2}{20}=1}$
7. $9(x^2-6x+9)-4(y^2-4y+4)=81-36-29+16=-81+...$ Regroup: $9(x-3)^2-4(y-2)^2=-29+81-16=36$; divide by 36: $\frac{(x-3)^2}{4}-\frac{(y-2)^2}{9}=1$. **Horizontal hyperbola**, center $(3,2)$

Chapter 9 Practice — Answers

ANSWER KEY

1. $d=4$; $a_{15}=3+14(4)=\mathbf{59}$; $S_{15}=\frac{15}{2}(3+59)=\frac{15}{2}(62)=\mathbf{465}$
2. $r=3$; $a_8=2\cdot 3^7=2\cdot 2187=\mathbf{4374}$; $S_8=2\cdot\frac{1-3^8}{1-3}=2\cdot\frac{-6560}{-2}=\mathbf{6560}$
3. $r=\frac{1}{3}<1$: **converges**; $S=\frac{5}{1-\frac{1}{3}}=\frac{5}{\frac{2}{3}}=\mathbf{\frac{15}{2}}$
4. $r=1.2>1$: **diverges** (no finite sum)
5. $a_2=3(2)-1=5$; $a_3=3(5)-1=14$; $a_4=3(14)-1=\mathbf{41}$
6. $3(1)+1+3(2)+1+\cdots+3(5)+1=4+7+10+13+16=\mathbf{50}$
7. *Proof:* Base: $n=1$: $\sum_{i=1}^{1} i=1=\frac{1(2)}{2}$ ✓. Hypothesis: assume $\sum_{i=1}^{k} i=\frac{k(k+1)}{2}$. Step: $\sum_{i=1}^{k+1} i=\frac{k(k+1)}{2}+(k+1)=(k+1)\left(\frac{k}{2}+1\right)=\frac{(k+1)(k+2)}{2}$. This is $P(k+1)$. ✓By induction, holds for all $n\geq 1$.
8. *Proof:* Base: $\sum_{i=1}^{1} 3^i=3=\frac{3(3-1)}{2}=3$ ✓. Hypothesis: $\sum_{i=1}^{k} 3^i=\frac{3(3^k-1)}{2}$. Step: $\sum_{i=1}^{k+1} 3^i=\frac{3(3^k-1)}{2}+3^{k+1}=\frac{3\cdot 3^k-3+2\cdot 3^{k+1}}{2}=\frac{3\cdot 3^k-3+6\cdot 3^k}{2}=\frac{9\cdot 3^k-3}{2}=\frac{3(3^{k+1}-1)}{2}$ ✓

Final Exam Mixed Practice — Answers

ANSWER KEY

1. $d = \sqrt{36+64} = \mathbf{10}$; $M = (\mathbf{-1}, \mathbf{5})$
2. $g(-1) = -1$; $(f \circ g)(-1) = f(-1) = 1 - 1 = \mathbf{0}$
3. Swap x and y: $x = \frac{y-3}{2y+1}$; solve: $x(2y+1) = y - 3 \Rightarrow 2xy - y = -3 - x \Rightarrow y(2x-1) = -(x+3)$; $f^{-1}(x) = \mathbf{\frac{-(x+3)}{2x-1}}$
4. Quadratic formula: $x = \frac{3 \pm \sqrt{9+72}}{4} = \frac{3 \pm 9}{4}$; $\mathbf{x = 3}$ and $\mathbf{x = -\frac{3}{2}}$
5. Perpendicular slope: $-\frac{3}{2}$; $y + 1 = -\frac{3}{2}(x-4) \Rightarrow \mathbf{y = -\frac{3}{2}x + 5}$
6. Vertex $(\mathbf{2}, \mathbf{-6})$; x-intercepts: $|x-2| = 2$, so $x = 4$ and $x = 0$; y-intercept: $f(0) = 3(2) - 6 = 0$
7. Factor by grouping: $x^2(x+1) - 9(x+1) = (x+1)(x^2-9) = (x+1)(x-3)(x+3)$. Zeros: $\mathbf{-1, 3, -3}$
8. Leading term $-2x^4$: both ends $\to -\infty$. Zeros: $-2x^2(x^2-4) = 0 \Rightarrow x = 0$ (mult.2), $x = \pm 2$ (mult.1)
9. Reflect over x-axis, vertical stretch by 3, shift left 2, shift up 1 (order matters in practice)
10. $\log_3 \frac{x+4}{x-2} = 2 \Rightarrow \frac{x+4}{x-2} = 9 \Rightarrow x + 4 = 9x - 18 \Rightarrow \mathbf{x = \frac{22}{8} = \frac{11}{4}}$
11. $180 = 200e^{10k} \Rightarrow k = \frac{\ln 0.9}{10} \approx -0.01054$; $50 = 200e^{kt} \Rightarrow t = \frac{\ln 0.25}{k} \approx \mathbf{131.5}$ years
12. $y = \log_5(3x-1) \Rightarrow$ swap: $x = \log_5(3y-1) \Rightarrow 5^x = 3y - 1 \Rightarrow f^{-1}(x) = \mathbf{\frac{5^x+1}{3}}$
13. Eliminate y: from eq.2 $y = 5x - 3$; sub: $2x + 3(5x-3) = 7 \Rightarrow 17x = 16 \Rightarrow x = \frac{16}{17}$, $y = \frac{29}{17}$. $(\mathbf{\frac{16}{17}}, \mathbf{\frac{29}{17}})$
14. Row reduce 3×3 system: $\mathbf{x = 1, y = 2, z = 3}$
15. $AB = \begin{bmatrix} 6 & 5 \\ 5 & -4 \end{bmatrix}$; $2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$; $AB - 2I = \begin{bmatrix} 4 & 5 \\ 5 & -6 \end{bmatrix}$
16. Let s=smaller, l=larger: $s + l = 320$ and $l = 2s + 40$. So $3s + 40 = 320$, $s = \mathbf{93.3}\ldots$ round: $s = 93$, $l = 227$.
17. **Ellipse**: center $(1, -2)$, $a = 4$ (horizontal), $b = 2$; $c = \sqrt{12} = 2\sqrt{3}$; foci $(1 \pm 2\sqrt{3}, -2)$
18. $r = \sqrt{(7-3)^2 + (-2+5)^2} = \sqrt{16+9} = 5$; $(x-3)^2 + (y+5)^2 = \mathbf{25}$
19. Vertex $(2, 4)$, focus $(2, 7)$: $k + \frac{1}{4a} = 7$ and $k = 4$, so $\frac{1}{4a} = 3$, $a = \frac{1}{12}$; equation: $y = \mathbf{\frac{1}{12}(x-2)^2 + 4}$
20. $a = 3$, $c = 5$, $b^2 = 16$; equation: $\frac{x^2}{9} - \frac{y^2}{16} = 1$; asymptotes: $y = \pm\frac{4}{3}x$
21. Complete square: $4(x^2 - 2x + 1) + (y^2 + 6y + 9) = 3 + 4 + 9 = 16$; $\frac{(x-1)^2}{4} + \frac{(y+3)^2}{16} = 1$. **Ellipse** (vertical major axis)

22. System: $a_1 + 2d = 7$ and $a_1 + 7d = 22$; $5d = 15$, $d = 3$, $a_1 = 1$; $S_{20} = \frac{20}{2}(2 + 59) = \mathbf{610}$
23. First drop: 16 ft. Total upward+downward after each bounce: $2{\cdot}16(\frac{3}{4}) + 2{\cdot}16(\frac{3}{4})^2 + \cdots =$

CLOSING: A NOTE FROM YOUR INSTRUCTOR

CHECK YOUR VOICE

You made it to the end of this study guide. Let's talk about what that means.

This guide covers nine chapters of college algebra. It contains:

- Every key formula in the book — in one place
- 65+ practice problems with complete solutions
- Exam strategies built around how your brain actually works under pressure
- Integration problems that connect every chapter to every other chapter

But here's what it doesn't contain, and what no textbook or study guide can give you:
The evidence you've built by doing the work.

That evidence is in your notebooks. In the problems you got wrong and tried again. In the moments you explained a concept to someone else and realized you actually understood it. In the exams you took and the homework you struggled through.

If your inner voice right now says: "I'm still not ready" —
Let me offer a different interpretation: that voice is the voice of someone who takes mathematics seriously. Someone who knows there's more to learn. Someone who has standards.
That is not the voice of a person who "can't do math."
That is the voice of a mathematician.

You're smarter than you think.
You always were. The math just proved it.

Quick Reference: Formula Card

Everything you need to know. One page. Tear it out. Put it on your desk.

Ch. 1–3: Foundations

Distance: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
Slope: $m = \frac{y_2-y_1}{x_2-x_1}$
Slope-intercept: $y = mx + b$
Vertex formula: $x = -\frac{b}{2a}$
Quadratic formula: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Discriminant: $\Delta = b^2 - 4ac$

Ch. 4–5: Polynomials & Transforms

Polynomial: $f(x) = a_n x^n + \cdots + a_0$
Factor Theorem: $f(r) = 0 \Leftrightarrow (x - r)$ factor
Transformations: $g(x) = af(b(x-h)) + k$
Composition: $(f \circ g)(x) = f(g(x))$
Inverse: swap x, y and solve for y

Ch. 6: Exponentials & Logs

Exp growth/decay: $A = A_0 e^{kt}$
Compound: $A = P(1 + \frac{r}{n})^{nt}$
Log def: $\log_b x = y \Leftrightarrow b^y = x$
Product: $\log(MN) = \log M + \log N$
Power: $\log(M^p) = p \log M$
Change of base: $\log_b x = \frac{\ln x}{\ln b}$

Ch. 7: Systems & Matrices

Matrix multiply: $(AB)_{ij} = \sum_k a_{ik} b_{kj}$
2×2 inverse: $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
Solve: row reduce augmented matrix $[A|B]$
Or: $X = A^{-1}B$ when $\det(A) \neq 0$

Ch. 8: Conics

Circle: $(x - h)^2 + (y - k)^2 = r^2$
Ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; $c^2 = a^2 - b^2$
Parabola: $y = a(x - h)^2 + k$; focus at $k + \frac{1}{4a}$
Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $c^2 = a^2 + b^2$
Asymptotes: $y = \pm\frac{b}{a}x$ (horizontal hyp.)

Ch. 9: Sequences & Series

Arith. term: $a_n = a_1 + (n - 1)d$
Geom. term: $a_n = a_1 r^{n-1}$
Arith. sum: $S_n = \frac{n}{2}(a_1 + a_n)$
Geom. sum: $S_n = a_1 \frac{1-r^n}{1-r}$
Infinite geom.: $S = \frac{a_1}{1-r}$, $|r| < 1$
Induction: Base + Hypothesis + Step

You made it.

Nine chapters. Three review chapters. One complete study guide.

You solved systems of equations, proved theorems with mathematical induction, read conic equations like stories, and built functions from scratch.

That is not what someone who "can't do math" does.

That is what you did.

You're smarter than you think.
You always were. The math just proved it.

Safaa Dabagh · Santa Monica College · West Los Angeles College · 2026